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ISBN 1-932661-80-8

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Contents

Chapter 5 Trigonometric Equations

Chapter 6 Oblique Triangles and Vectors

Chapter 7 Complex Numbers and Polar Coordinates


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CHAPTER 3

Properties of the Trigonometric Functions

To understand many applications of the trigonometric functions it is necessary to have a good understanding of the properties of these functions. As with any functions, we can gain a great deal of knowledge from their graphs. There are many graphing calculators and computer programs available today that are capable of graphing functions defined from equations. We will illustrate the graphing of trigonometric functions by two methods:

1. obtaining important information about the graph, then using this information to sketch the graph by hand, and
2. using a graphing calculator.

We use the TI-81 graphing calculator to illustrate using a graphing calculator. The process is practically the same with another brand or model.

Before going further, we show some basics of using the TI-81 graphing calculator. The reader can jump to section 3–1 if not using a graphing calculator.

3–0 TI-81 graphing basics

Setting the range for the screen

RANGE
Xmin=−10
Xmax=10
Xscl=1
Ymin=−10
Ymax=10
Yscl=1
Xres=1

Table 3–1

Graphing calculators have a way to describe which part of the coordinate plane will be displayed. It is called setting the RANGE. Using the [RANGE] key shows a display similar to that in table 3–1. The Xmin and Xmax values refer to the range of x values which will be displayed. The Ymin and Ymax values refer to the range of y values which will be displayed. The Xscl and Yscl values refer to the tick marks which will appear on the screen. The Xres refers to the number of x values which will be calculated. It should be left at 1.

Throughout the text we will show the Xmin, Xmax, Xscl, Ymin, Ymax and Yscl values, in this order, in a box labeled RANGE. For the values shown in figure 3–1 we would write [RANGE −10,10,1,−10,10,1]. This omits the value for Xres, which we will assume is 1.

By entering numeric values and using the $\boxed{\text{ENTER}}$ key to move down the list, the values in the RANGE can be changed. Note that to obtain a negative number the $\boxed{(-)}$ (change sign) key is used, not the $\boxed{-}$ (subtract) key.

Figure 3–1 shows the screen appearance for various settings of Xmax, Xmin, Ymax, and Ymin. Xscl and Yscl are 1 except where labeled Yscl=3 and Xscl=2. After setting these values with the $\boxed{\text{RANGE}}$ key, use the $\boxed{\text{GRAPH}}$ key to show the screen. Using the $\boxed{\text{CLEAR}}$ button readies the calculator for numeric calculations again. The settings in part (a) of figure 3–1 are the "standard" settings, obtained by selecting $\boxed{\text{ZOOM}}$ 6.

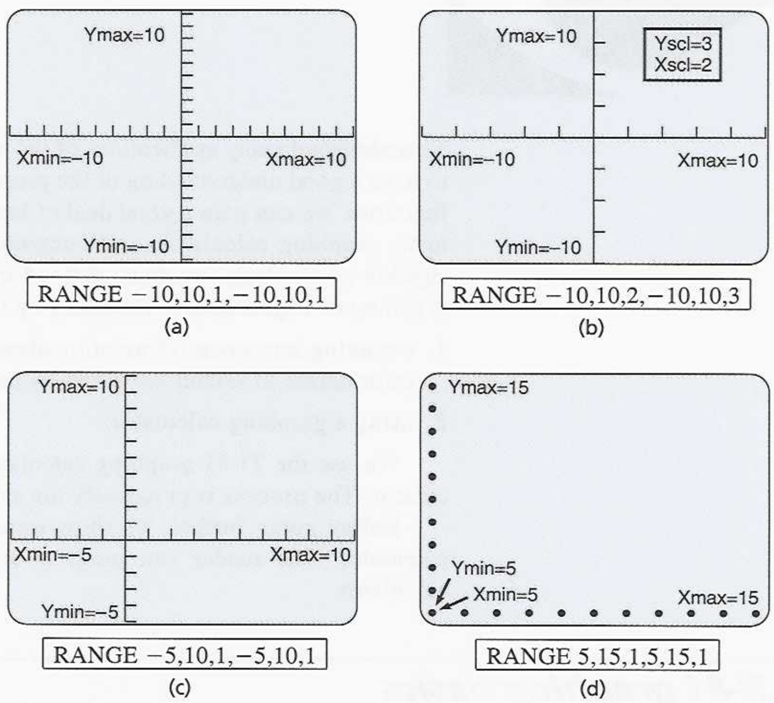


Figure 3–1

Observe that the distance between units are not the same on the screen. The calculator automatically makes horizontal units 1.5 times as long as vertical units. To have horizontal and vertical distances the same, use the $\boxed{\text{ZOOM}}$ function, where option 5 says SQUARE. This makes the screen use the same scale for distance vertically and horizontally by changing the values of Xmin and Xmax. In most cases, having equal horizontal and vertical scales will not be important.

When graphing trigonometric functions [ZOOM] 7 (Trig) can be useful. It sets Range settings to

$$\boxed{\text{RANGE } -6.28\ (-2\pi), 6.28\ (2\pi), 1.57 \left(\frac{\pi}{2}\right), -3, 3, .25}\ .$$

Graphing an equation in which y is described in terms of x

If an equation describes values for a variable y in terms of a variable x, the graphing calculator can be used to view the graph of the equation.

■ ***Example 3–0 A***

Graph each equation.

1. Graph $y = 2x - 3$

x	$y\ (2x - 3)$
-3	-9
-2	-7
-1	-5
0	-3
1	-1
2	1
3	3

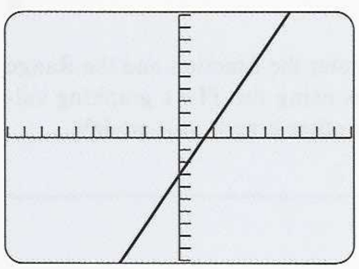
This could be done without a graphing calculator with practically no knowledge of graphing by a table of values, by letting x take on many values, such as -3, -2, -1, 0, 1, 2, 3, etc., and computing y for each one. In fact, this table is shown here. The y-values are computed by computing $2x - 3$ for the given x-value. Each pair of values for x and y represents an ordered pair (x,y) (we always write the x-value first). If we plot enough of these values in a coordinate system we start to see a picture emerge. In this case it is a straight line.

Of course the point of this section is to have the calculator automatically calculate the x- and y-values and plot them. Assuming the standard RANGE settings (obtained by [ZOOM] 6) proceed as follows to obtain the graph:

[Y=]	Allows us to enter up to four equations.
[2]	
[X|T]	The variable x.
[−]	
[3]	The display looks like

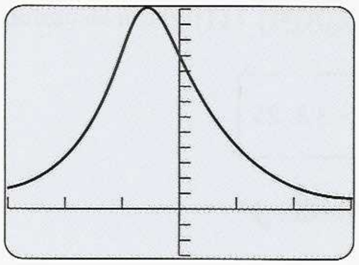
```
:Y1=2X−3
:Y2=
:Y3=
:Y4=
```

[GRAPH]



[Y=] 2 [X|T] [−] 3,
[RANGE −10,10,1,−10,10,1]

Note If there are any equations already entered for Y_1 use the [CLEAR] key before entering the equation. If there are any extra equations entered for Y_2, Y_3, or Y_4, move down with the down arrow key [▽] to that equation and use the [CLEAR] key to clear that entry. Use [ZOOM] 6 to obtain the standard Range settings. The figure shows what the display will look like.



Y= (X|T x^2 +
X|T + 1) x^{-1},
RANGE −3,3,1,−.3,1.3,.1

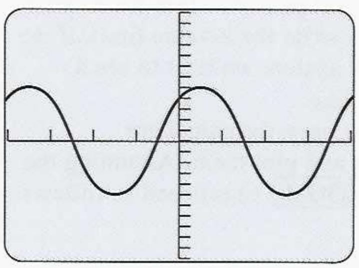
2. Graph $y = \dfrac{1}{x^2 + x + 1}$

The following steps would produce a graph similar to that shown in the figure.

Steps	Explanation
RANGE	Enter the x- and y-axis limits.
(−) 3 ENTER	Xmin becomes −3.
3 ENTER	Xmax becomes 3.
1 ENTER	Xscl becomes 1.
(−) .3 ENTER	Ymin becomes −0.3.
1.3 ENTER	Ymax becomes 1.3.
.1 ENTER	Yscl becomes 0.1.

The x^{-1} key is used to define a reciprocal (something divided into one).

Y= (X|T x^2 + X|T + 1) x^{-1} GRAPH



Y= SIN X|T +
COS X|T
RANGE −6.28,6.28,1.57,−3,3,.25

3. $y = \sin x + \cos x$

Make sure the calculator is in radian mode (with the MODE key).

Steps	Explanation
Y= CLEAR SIN X|T + COS X|T	Enter graphing mode. Remove the previous function.
ZOOM 7	Select standard settings for trigonometric functions. The graphing begins automatically after ZOOM 7 is selected.

In the remainder of this text we show how to enter the function and the Range settings for each graph, assuming the reader is using the TI-81 graphing calculator. The steps are practically the same for other brands and models.

3–1 Graphs and properties of the sine, cosine, and tangent functions

Graph of the sine function

The graph of $y = \sin x$ for x between 0 and 2π is shown in figure 3–2. This graph can be obtained by plotting points for various values of x. For example, the points for the following table of values are shown in the figure.

x (radians)	$\sin x$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$ (0.7)
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$ (0.9)
$\frac{\pi}{2}$	1

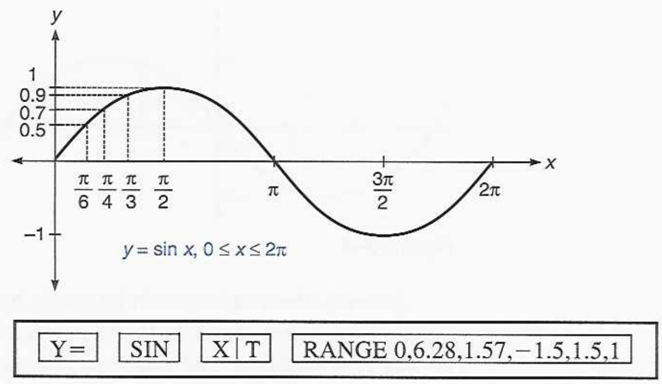


Figure 3–2

The graph in figure 3–2 repeats itself, as shown in figure 3–3, for other values of x because for values of x greater than 2π or less than 0 we have angles that are coterminal with values we have already plotted. Thus, every 2π units we find that the part of the graph that was shown in figure 3–3 is repeated. If we memorize the graph in figure 3–2, we can use it to reproduce the graph in figure 3–3.

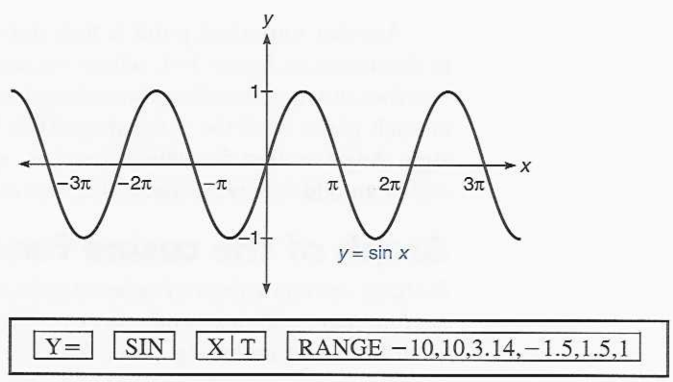


Figure 3–3

The repetitious nature of the sine function can be described with the identity

$$\sin x = \sin (x + k \cdot 2\pi),\ k \text{ any integer}$$

We say that *the sine function is 2π-periodic,* or is periodic with period 2π.

Note Any function that repeats the same pattern over and over is said to be **periodic.** The period is the length of the shortest pattern that produces the function when repeated. Algebraically, a function f is *p-periodic* if there is a number p, $p > 0$, such that

$$f(x + p) = f(x)$$

for all x in the domain, and p is the smallest such number.

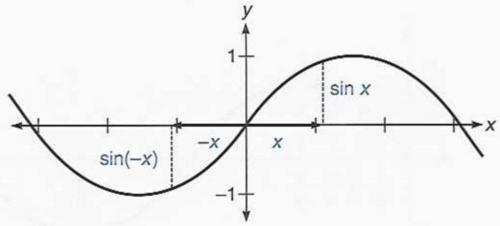


Figure 3–4

Several observations can be made by looking at the graph of $y = \sin x$ in figure 3–4. The domain of the sine function is all real numbers; that is, x can be any real number. The range of the sine function is restricted to the values between and including -1 to 1. In other words, if R represents the collection of all real numbers, then:

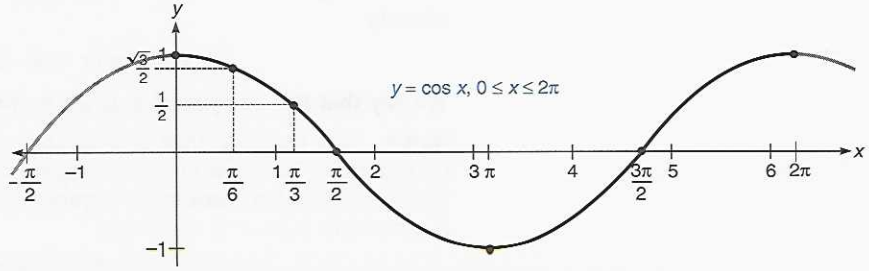
$$\text{Domain}_{\text{sine}}: R$$
and
$$\text{Range}_{\text{sine}}: -1 \leq y \leq 1$$

Note The domain is verbalized as "all real numbers"; the range is verbalized as "all real numbers y having the property that $y \geq -1$ and $y \leq 1$."

Another important point is that $\sin(-x) = -(\sin x)$ for any value x. This is illustrated in figure 3–4, where we see that if we go equal distances in the positive and negative directions along the x-axis, the value of the sine function at each place is of the same magnitude (absolute value) but of the opposite sign. Any function for which $f(-x) = -f(x)$ is true for all x in its domain is called an odd function; therefore, *sine is an odd function.*

Graph of the cosine function

Plotting various values of ordered pairs (x,y), where $y = \cos x$, and then connecting them with a smooth curve produces the graph shown in figure 3–5 for x between and including 0 and 2π. For example, we know that $\cos 0 = 1$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \pi = -1$, etc. These values are shown in figure 3–5.

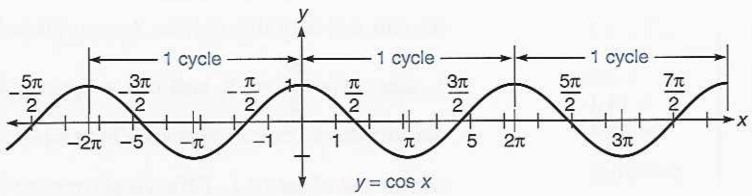


Y=	COS	X|T	RANGE 0,6.28,1.57,−1.5,1.5,1

Figure 3–5

Just as with the sine function, the cosine function repeats after 2π units. The identity that states this algebraically is $\cos x = \cos(x + k \cdot 2\pi)$, k any integer. As with the sine function, we also say that *the cosine function is 2π-periodic.*

The graph of $y = \cos x$ is shown in figure 3–5. If we memorize the graph in figure 3–5, we can use it to reproduce the graph in figure 3–6.



Y=	COS	X|T	RANGE −10,10,3.14,−1.5,1.5,1

Figure 3–6

Several observations can be made by looking at the graph of $y = \cos x$ in figure 3–6. The domain of the cosine function is all real numbers; that is, x can be any real number. The range of the cosine function is restricted to the values between and including -1 to 1. These are, of course, the same as the domain and range of the sine function.

$$\text{Domain}_{\text{cosine}}: R$$
and
$$\text{Range}_{\text{cosine}}: -1 \leq y \leq 1$$

We also see that $\cos(-x) = \cos x$. This is illustrated in figure 3–7, where we see that if we go equal distances in the positive and negative directions along the x-axis, the value of the cosine function at each place is the same value. Any function for which $f(-x) = f(x)$ is true for all x in its domain is called an even function; therefore, *cosine is an even function.*

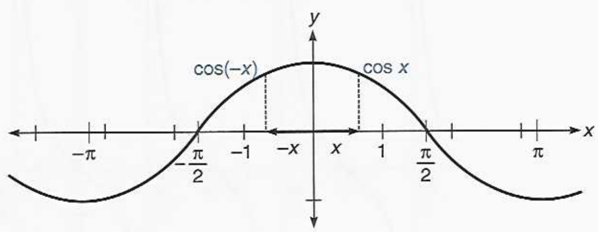


Figure 3–7

Finally, *the graphs of the sine and cosine functions have exactly the same shape;* either one becomes the other if it is shifted right or left a suitable amount. The smallest such amount is $\frac{\pi}{2}$, which is described in the statement that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. This statement is proved in chapter 5.

x	$\tan x$
0	0
$\frac{\pi}{6} \approx 0.52$	$\frac{\sqrt{3}}{3} \approx 0.6$
$\frac{\pi}{4} \approx 0.79$	1
$\frac{\pi}{3} \approx 1.05$	$\sqrt{3} \approx 1.7$
1.25	≈ 3.0
1.50	≈ 14.1
1.55	≈ 48.1
$\frac{\pi}{2} \approx 1.57$	undefined

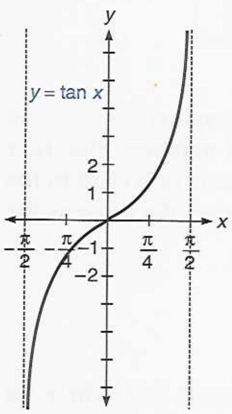
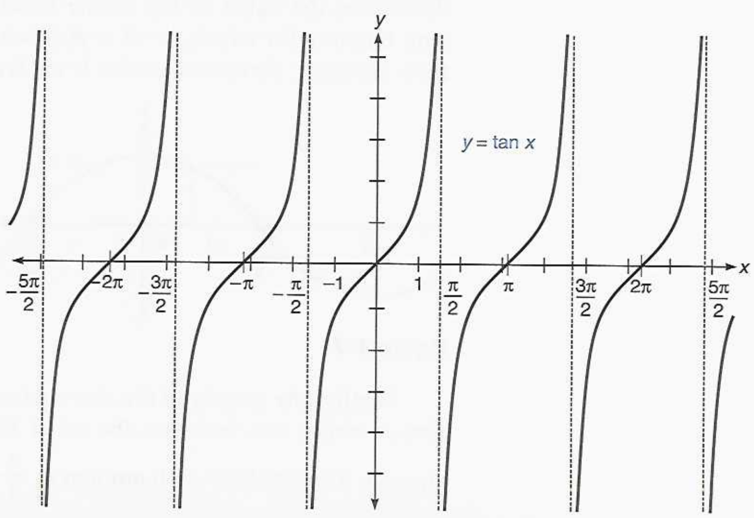


Figure 3–8

Graph of the tangent function

To obtain the graph of the tangent function, we also compute values and plot points. Some values for x between 0 and $\frac{\pi}{2}$ are shown in the table. Observe that as x gets closer to $\frac{\pi}{2}$, $\tan x$ gets larger. If we recall that $\tan x = \frac{\sin x}{\cos x}$, we can see why this is true. As x approaches $\frac{\pi}{2}$ (from below), $\sin x$ approaches 1, since $\sin \frac{\pi}{2} = 1$, and $\cos x$ approaches 0, since $\cos \frac{\pi}{2} = 0$. Now as the denominator, $\cos x$, gets smaller and smaller we divide it into values of $\sin x$, which are close to 1. Effectively we are calculating $\frac{1}{\cos x}$, or the reciprocal of $\cos x$. The smaller the absolute value of a number, the larger is its reciprocal. For example, the reciprocal of $\frac{1}{100}$ is 100, and of $\frac{1}{10{,}000}$ is 10,000. Thus, as $\cos x$ gets smaller, $\frac{1}{\cos x}$ gets larger and larger, and so $\tan x = \frac{\sin x}{\cos x}$ gets larger and larger. Since $\cos \frac{\pi}{2} = 0$, $\tan \frac{\pi}{2}$ is not defined. For the same reason, $\tan\left(-\frac{\pi}{2}\right)$ is not defined either. The graph of $y = \tan x$ is shown for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ in figure 3–8. It can be proved that *the tangent function is π-periodic;* that is, for any x, $\tan x = \tan(x + k\pi)$, k any integer. The actual proof will be an exercise in chapter 5. This π-periodic property means that the graph of the tangent function repeats every π units. More of the graph of $y = \tan x$ is shown in figure 3–9. The vertical dashed lines indicate values



Y= TAN X|T RANGE −7,7,1.57,−5,5,1

Figure 3–9

of x for which the tangent function is not defined. Since $\tan x = \dfrac{\sin x}{\cos x}$, we know that the tangent function is not defined wherever $\cos x = 0$. The vertical dashed lines are called **asymptotes** of the tangent function and occur wherever $\cos x = 0$.

The domain of the tangent function is all values of x except where $\cos x = 0$, and the range is all values of y.

$$\text{Domain}_{\text{tangent}}: x \neq \frac{\pi}{2} + k\pi, k \text{ any integer}$$

$$\text{Range}_{\text{tangent}}: R$$

It will also be an exercise to show that *the tangent function is an odd function;* that is, $\tan(-x) = -(\tan x)$ for any x in its domain.

Table 3–2 summarizes the properties of the three functions we have examined.

Function	**Domain**	**Range**	**Period**
$y = \sin x$	R	$-1 \leq y \leq 1$	2π
$y = \cos x$	R	$-1 \leq y \leq 1$	2π
$y = \tan x$	$x \neq \frac{\pi}{2} + k\pi$	R	π

$\sin(-x) = -(\sin x)$ (odd)
$\cos(-x) = \cos x$ (even)
$\tan(-x) = -(\tan x)$ (odd)

Table 3–2

We can use the odd-even properties of these functions to simplify some computations.

■ ***Example 3–1 A***

1. Find $\tan\left(-\dfrac{5\pi}{6}\right)$.

Since tangent is an odd function, we know that

$$\tan\left(-\frac{5\pi}{6}\right) = -\left(\tan\frac{5\pi}{6}\right)$$

$\dfrac{5\pi}{6}$ terminates in quadrant II, so its reference angle is $\pi - \dfrac{5\pi}{6} = \dfrac{\pi}{6}$, and $\tan\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{3}$. Also, the tangent function is negative in quadrant II, so $\tan\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{3}$.

Therefore, $-\left(\tan\dfrac{5\pi}{6}\right) = -\left(-\dfrac{\sqrt{3}}{3}\right) = \dfrac{\sqrt{3}}{3}$ and $\tan\left(-\dfrac{5\pi}{6}\right) = \dfrac{\sqrt{3}}{3}$.

2. Find $\cos(-210°)$.

Since the cosine function is an even function, $\cos(-210°) = \cos 210°$. The reference angle for $210°$ is $30°$, and $\cos 30° = \frac{\sqrt{3}}{2}$. Since $210°$ terminates in quadrant III where the cosine function is negative, $\cos 210° = -\frac{\sqrt{3}}{2}$. Therefore, $\cos(-210°) = -\frac{\sqrt{3}}{2}$. ■

Mastery points

Can you

- Sketch the graphs of the sine, cosine, and tangent functions?
- State the domain, range, and period of the sine, cosine, and tangent functions?
- Use the odd-even properties to compute the values of $\sin x$, $\cos x$, and $\tan x$ for negative values of x?

Exercise 3–1

1. Sketch the graphs of
   a. $y = \sin x$ **b.** $y = \cos x$ **c.** $y = \tan x$
2. From memory, or using their graphs as an aid, state the domain, range, and period of each of the functions sine, cosine, and tangent.
3. Using the graph of $y = \sin x$ as a guide, describe all values of x for which $\sin x$ is
   a. 1 **b.** -1 **c.** 0
4. Using the graph of $y = \cos x$ as a guide, describe all values of x for which $\cos x$ is
   a. 1 **b.** -1 **c.** 0
5. Using the graph of $y = \tan x$ as a guide, describe all values of x for which $\tan x$ is 0.

Use the appropriate property, odd or even, to simplify the computation of the exact value of the (a) sine, (b) cosine, and (c) tangent functions for the following values.

6. $-\frac{\pi}{3}$ 7. $-\frac{\pi}{6}$ 8. $-45°$ 9. $-\frac{5\pi}{3}$

In the text we stated that a function f is odd if $f(-x) = -f(x)$ for all x in its domain and is even if $f(-x) = f(x)$ for all x in its domain. An algebraic example of an odd function is $f(x) = x^3$, since

$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x) \end{aligned}$$

Thus, to illustrate this point, again using $f(x) = x^3$, we can see that $f(-2) = -8$, $f(2) = 8$, and so $f(-2) = -f(2)$.

An example of an even function is $f(x) = x^2$, since we can show that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= (-x)^2 \\ &= x^2 \\ &= f(x) \end{aligned}$$

Some functions are neither odd nor even, such as $f(x) = x - 3$, since $f(-x) = -x - 3$, but $-f(x) = -(x - 3) = -x + 3$, so $f(-x)$ is neither $f(x)$ nor $-f(x)$, as we see when we compare

$$\begin{aligned} f(x) &= x - 3 \\ f(-x) &= -x - 3 \\ -f(x) &= -x + 3 \end{aligned}$$

Compute $f(-x)$ and $-f(x)$ for each of the following functions, and state whether the function is odd, even, or neither.

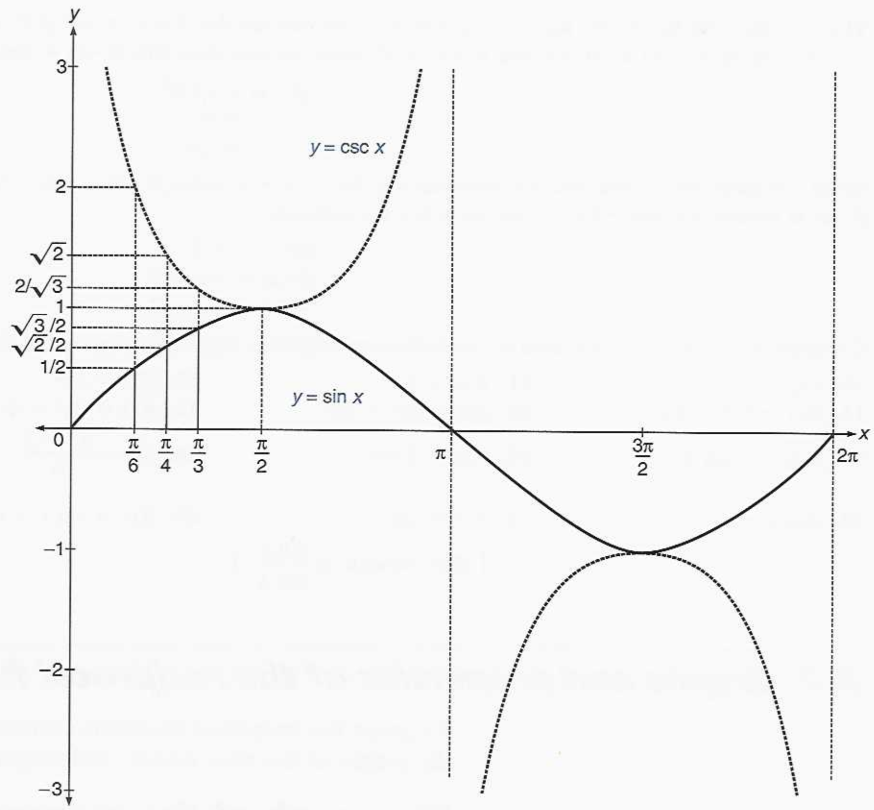
10. $f(x) = x$
11. $f(x) = 3x$
12. $f(x) = 3x^2$
13. $f(x) = -x^2$
14. $f(x) = 2x^4 - 4x^2$
15. $f(x) = 3x^2 - 2x^4$
16. $f(x) = 2x^3 - 4x$
17. $f(x) = 3x - 2x^3$
18. $f(x) = 3 \sin x$
19. $f(x) = 2 \cos x$
20. $f(x) = \dfrac{x^2 - 1}{4}$
21. $f(x) = \dfrac{x^5 - x^3}{x}$
22. $f(x) = \sin^2 x$
23. $f(x) = \tan x$ $\left(\textit{Hint: } \text{Rewrite as } \dfrac{\sin x}{\cos x}.\right)$
24. $f(x) = \sin x + \cos x$
25. $f(x) = \dfrac{\sin x}{x}$

3–2 Graphs and properties of the reciprocal functions

To graph the reciprocal functions cosecant, secant, and cotangent, we can use the graphs of the sine, cosine, and tangent functions as our guide.

The graph of the cosecant function

Remember that $\csc x = \dfrac{1}{\sin x}$. Thus, to graph $y = \csc x$ we first graph $y = \sin x$, and then examine the reciprocal values. Figure 3–10 shows the graph of $y = \sin x$ for $0 \le x \le 2\pi$, as well as dashed lines that represent the cosecant function values for these values of x.



$\boxed{\text{Y=}}$ $\boxed{(}$ $\boxed{\text{SIN}}$ $\boxed{\text{X|T}}$ $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{Y=}}$ $\boxed{\text{SIN}}$ $\boxed{\text{X|T}}$

$\boxed{\text{RANGE } 0,6.28,1.57,-3,3,1}$

Figure 3–10

x	$\sin x$	$\csc x$	
$\frac{\pi}{6}$	$\frac{1}{2}$	2	
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	(1.4)
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$	(1.2)
$\frac{\pi}{2}$	1	1	

To see that the dashed lines represent the values for the cosecant, consider the table, which shows both the sine and cosecant values for selected values of x. These selected values are also illustrated in figure 3–10.

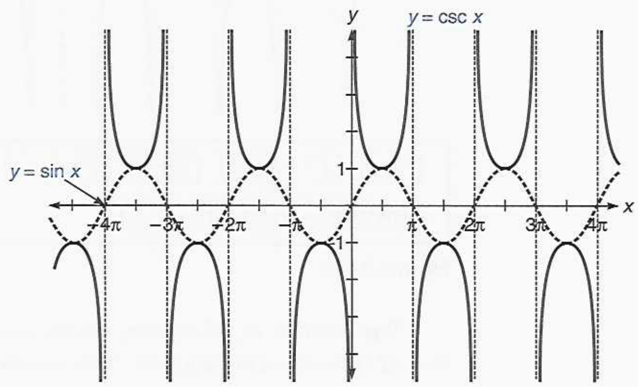
Referring to figure 3–10 and the table, we see that as x increases from $\frac{\pi}{6}$ to $\frac{\pi}{2}$, $\sin x$ increases from $\frac{1}{2}$ to 1, and the reciprocal values, $\csc x$, decrease from 2 to 1. Also, as $\sin x$ decreases in absolute value (i.e., gets closer to the x-axis), the reciprocal gets larger in absolute value. Wherever $\sin x$ is 1 or -1, so is its reciprocal value, $\csc x$. Wherever $\sin x$ approaches 0, the absolute value of $\csc x$ approaches infinity (gets larger and larger). Note that if $\sin x$ approaches 0 through positive values, its reciprocal gets larger and larger, and

wherever $\sin x$ approaches 0 through negative values, $\csc x$ becomes larger and larger in *absolute value,* although it is negative.

To graph $y = \csc x$, we can rely on the graph of $y = \sin x$ for our guide. The steps are:

1. Graph $y = \sin x$.
2. Wherever $\sin x$ is $+1$ or -1, so is $\csc x$.
3. Wherever $\sin x$ is 0, draw vertical dashed lines (asymptotes).
4. As $\sin x$ approaches 0, draw $\csc x$ getting greater and greater in absolute value, positive or negative depending on the sign of the sine function.

The graph of $y = \csc x$ is shown in figure 3–11; the graph of $y = \sin x$ is shown as a dashed line.



Y=	(	SIN	X|T	)	x^{-1}

RANGE −10,10,3.14,−5,5,1

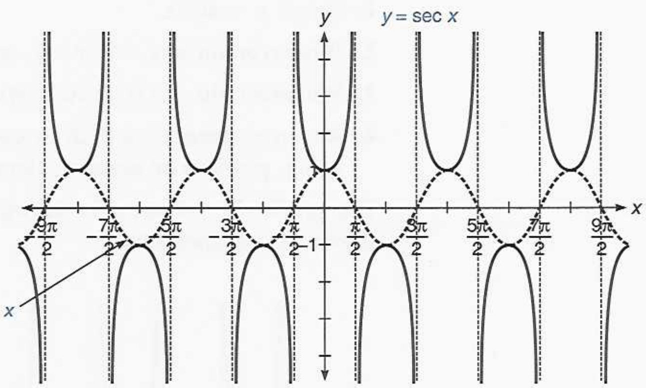
Figure 3–11

Note that the domain of the cosecant function is all x except where $\sin x = 0$, and the range is all y greater than or equal to 1 in absolute value. This range reflects the fact that since $\sin x$ is less than or equal to 1 in absolute value, $\dfrac{1}{\sin x}$ must be greater than or equal to 1 in absolute value. Also, the cosecant function is 2π-periodic, just as the sine function is.

The graph of the secant function

The graph of $y = \sec x$ is analyzed in the same manner as the graph of $y = \csc x$, except that we are considering $y = \dfrac{1}{\cos x}$ instead of $y = \dfrac{1}{\sin x}$.

Thus, to graph $y = \sec x$, we can rely on the graph of $y = \cos x$ for our guide. The steps are the same as when graphing the cosecant function, except that the cosine function is our guide. This produces the graph of $y = \sec x$, which is shown along with the graph of $y = \cos x$ in figure 3–12.



Y=	(	COS	X|T	)	x^{-1}
RANGE −10,10,3.14,−5,5,1					

Figure 3–12

The domain is, of course, where $\cos x \neq 0$, and the range is the same as that of the cosecant function. The secant function is also 2π-periodic.

The graph of the cotangent function

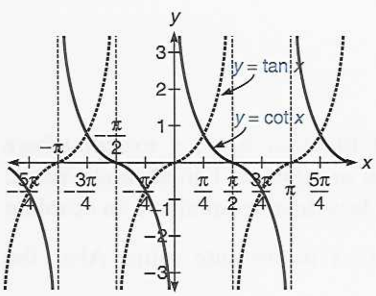


Figure 3–13

Since $\cot x = \dfrac{1}{\tan x}$ except where $\tan x = 0$, we can obtain the graph of $y = \cot x$ by analyzing the graph of $y = \tan x$ as we did previously for the other reciprocal functions. The graphs of both $y = \tan x$ and $y = \cot x$ are shown in figure 3–13. Note that wherever $\tan x$ is 0, we have a vertical asymptote, and wherever $\tan x$ approaches infinity or negative infinity, $\cot x$ approaches 0. This should make sense, since as a quantity gets greater and greater in absolute value, its reciprocal will get smaller and smaller in absolute value. Note that $y = \cot x$ is π-periodic as is the tangent function, its domain is all x except where $\sin x = 0$ $\left(\text{since } \cot x = \dfrac{\cos x}{\sin x}\right)$, and its range is all real numbers, as is the range of the tangent function.

Note One way to remember where the vertical asymptotes are for the cotangent function is to sketch the graph of the sine function. Wherever $\sin x$ is 0, $\cot x$ does not exist, and instead "goes to infinity or negative infinity." This is where we draw the vertical asymptotes.

The graph of $y = \cot x$ is shown in figure 3–14.

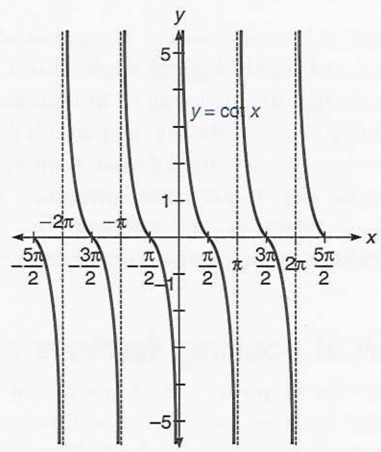


Figure 3–14

Y=	(	TAN	X|T	)
x^{-1}	ENTER			
RANGE −8,8,3.14,−5,5,1				

The properties of the three reciprocal trigonometric functions are summarized in table 3–3, where k is any integer.

Function	**Domain**	**Range**	**Period**
$y = \csc x$	$x \neq k\pi$	$\lvert y\rvert \geq 1$	2π
$y = \sec x$	$x \neq \frac{\pi}{2} + k\pi$	$\lvert y\rvert \geq 1$	2π
$y = \cot x$	$x \neq k\pi$	R	π

Table 3–3

Mastery points

Can you

- Sketch the graphs of the reciprocal functions?
- State the domain, range, and period of the reciprocal functions?

Exercise 3–2

1. Sketch the graphs of the three reciprocal trigonometric functions.

2. State the domain, range, and period for each of the three reciprocal trigonometric functions.

3. Use the identity $\csc x = \dfrac{1}{\sin x}$ to show that cosecant is an odd function.

4. Use the identity $\sec x = \dfrac{1}{\cos x}$ to show that secant is an even function.

5. Use the identity $\cot x = \dfrac{1}{\tan x}$ to show that cotangent is an odd function.

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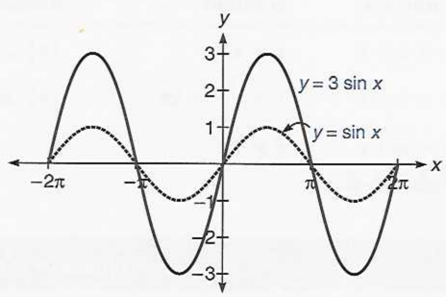
3–3 Linear transformations of the sine and cosine functions

In section 3–1 we developed the graphs of the sine and cosine functions. Most scientific and technological applications of these functions require that they be transformed in some way to fit measured or theoretical data. In this section we examine some of the ways in which this can be done. In particular, we will examine operations called **linear transformations.**

Graphically, linear transformations are operations that move a graph in some fixed direction, or ''squeeze'' or ''expand'' the graph uniformly. The linear transformations we examine are scaling factors and translations.

Vertical scaling factors and translations

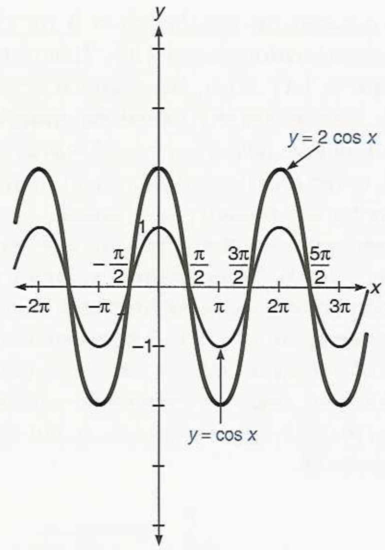
Consider the graph of $y = 3 \sin x$, and what this equation tells us to do to compute y for a given value of x. First, we are to compute the value of $\sin x$, and then multiply this value by 3. Thus, for the same value of x, the expression $3 \sin x$ will be 3 times greater (in absolute value) than the expression $\sin x$. If we then compare the graph of $y = 3 \sin x$ with the graph of $y = \sin x$, the y-values of the first must be 3 times greater than the y-values of the second. See figure 3–15.



[Y=] [SIN] [X|T] [ENTER]
3 [SIN] [X|T],
[RANGE −6.28,6.28,1.57,−4,4,1]

Figure 3–15

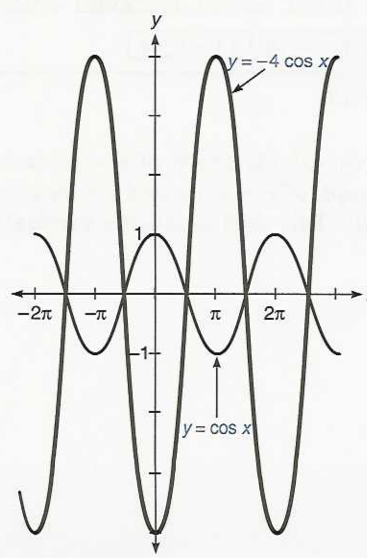
For the same reasons, the graph of $y = 2 \cos x$ is the same as that of $y = \cos x$, except that it reaches a magnitude of 2 instead of 1. See figure 3–16. Except for this vertical change in scale, the graph is the same as that of $y = \cos x$.



Y= COS X|T ENTER 2
COS X|T ,
RANGE −3.25,8,1.57,−3,3,1

Figure 3–16

If the coefficient of the sine or cosine value is negative, a reflection about the horizontal axis occurs. Consider, for example, the graph of $y = -4 \cos x$ compared to the graph of $y = \cos x$. To compute a y-value in $y = -4 \cos x$, we first compute $\cos x$ and then multiply this value by -4. Multiplying by a negative value changes the sign of the value being multiplied. Thus, whenever y is negative in the graph $y = \cos x$, the y in $y = -4 \cos x$ is positive and scaled (multiplied) by a factor of 4. Also, whenever y is positive in the graph $y = \cos x$, the y in $y = -4 \cos x$ is negative and scaled by a factor of 4. See figure 3–17.

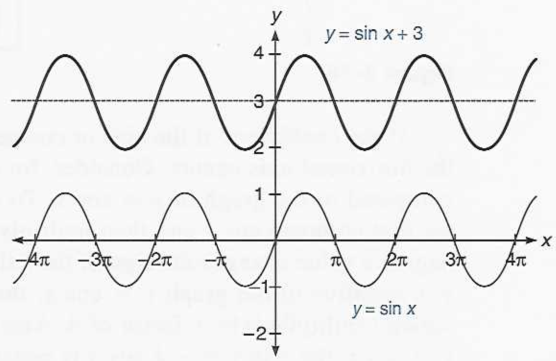


Y= COS X|T ENTER
(−) 4 COS X|T ,
RANGE −3.25,8,1.57,−5,5,1

Figure 3–17

In general, the graphs of $y = A \sin x$ and $y = A \cos x$ are scaled vertically by a vertical scaling factor $|A|$. That is, the magnitude of the graph is changed from one to $|A|$. *Also, the graph is reflected about the horizontal (x) axis if $A < 0$.* $|A|$ is usually called the **amplitude** of the function. If the sine or cosine function describes sound waves in the air, then the amplitude corresponds to the loudness of the sound; indeed, we often use the word amplitude to describe this property of a sound.

Now consider the graph of $y = \sin x + 3$ (not to be confused with $y = \sin(x + 3)$). To compute a value of y for a given x in $y = \sin x + 3$, we first compute the value of $\sin x$ and then add 3 to this value. This means that for a given x, $\sin x + 3$ is 3 units greater than $\sin x$. Thus, if we compare the graphs of $y = \sin x$ and $y = \sin x + 3$, the second graph must be 3 units higher than the first, since to compute a y in the second equation we do the same thing as in the first (compute $\sin x$), but then add 3. This vertical shift is shown in figure 3–18.



Y=	SIN	X|T	ENTER	SIN	X|T	+	3,
RANGE −4,10,3.14,−2,5,1							

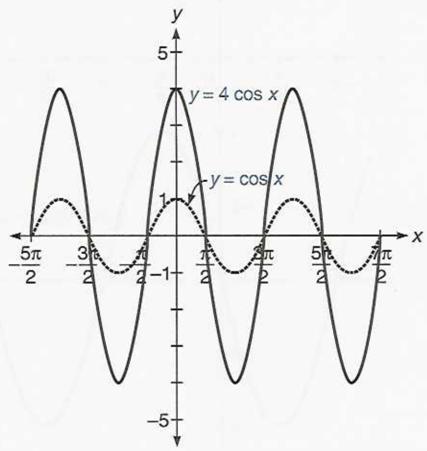
Figure 3–18

In general, the graph of $y = \sin x + D$ and $y = \cos x + D$, is the same as the graph of $y = \sin x$ and $y = \cos x$, respectively, but shifted up or down $|D|$ units. This shift is called a **vertical translation.**

■ ***Example 3–3 A***

1. Graph $y = 4 \cos x$.

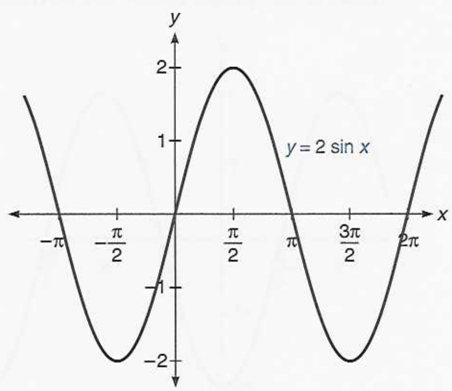
This is the same as the graph of $y = \cos x$, except scaled vertically by a factor of 4. Thus, the amplitude is 4. This is shown in the figure.



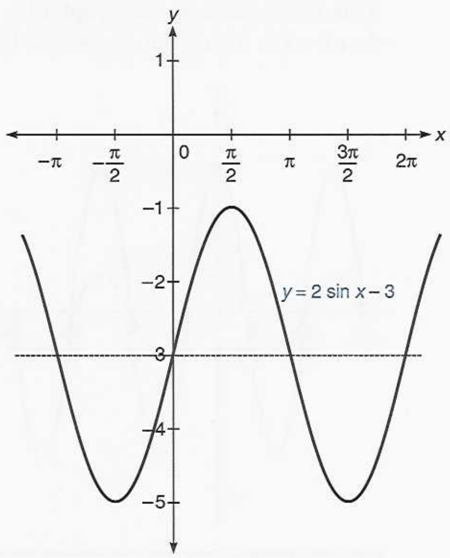
[Y=] 4 [COS] [X|T], [RANGE −8,12,1.57,−5,5,1]

2. Graph $y = 2 \sin x - 3$.

The 2 affects the amplitude of the graph, and the -3 causes a vertical translation down, because we are subtracting values from $2 \sin x$. In the first figure we draw a sine curve with amplitude 2.



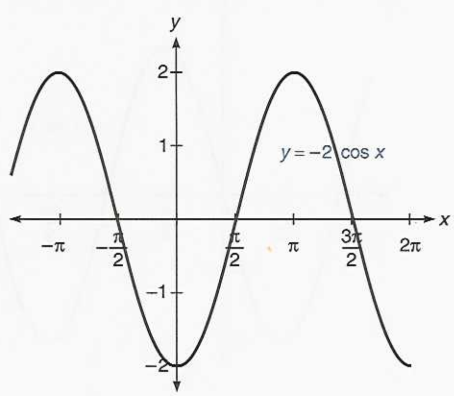
In the second figure we show the same curve translated down 3 units.



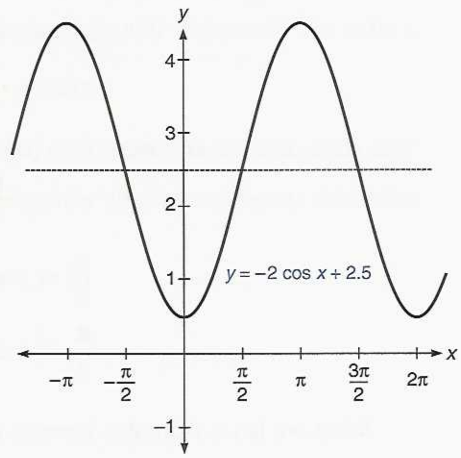
$\boxed{\text{Y=}}$ 2 $\boxed{\text{SIN}}$ $\boxed{\text{X\|T}}$ $\boxed{-}$ 3 $\boxed{\text{ENTER}}$ $\boxed{(-)}$ 3, $\boxed{\text{RANGE } -4,7,1.57,-6,1,1}$

3. Graph $y = -2 \cos x + 2.5$.

We first graph $y = -2 \cos x$; the amplitude is $|-2| = 2$, and the -2 reflects the graph about the horizontal axis. See the first figure.



Now we shift this graph up 2.5 units. See the second figure.



$\boxed{\text{Y=}}$ $\boxed{(-)}$ 2 $\boxed{\text{COS}}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 2.5 $\boxed{\text{ENTER}}$ 2.5,
$\boxed{\text{RANGE } -4,7,1.57,-1,5,1}$

Horizontal scaling factors and translations

We have seen how to transform the graph of a sine or cosine function vertically. It is just as important to be able to do this horizontally.

The **argument** of a function is the expression to be used as the domain element when doing computations. In $y = \sin x$ or $y = \cos x$, the x is the argument of the function. In $y = \sin 3x$, the expression $3x$ is the argument. In $y = \cos(x - 4)$ the expression $x - 4$ is the argument. In $y = 2 \sin 4x - 3$, $4x$ is the argument. The argument is the quantity we "take the sine or cosine of."

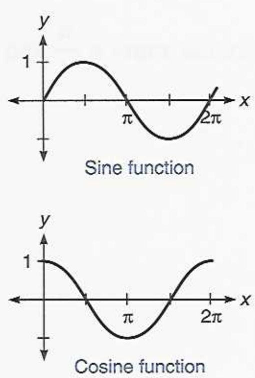


Figure 3–19

Now consider what we know about the sine and cosine functions. As the argument goes from 0 to 2π, each of these functions produces the graph shown in figure 3–19. We call the portion of each graph shown in figure 3–19 the **basic sine cycle** and the **basic cosine cycle,** respectively. Each of these basic cycles is repeated over and over to get the final, complete graphs of $y = \sin x$ and $y = \cos x$. The important fact is that *as the argument takes on values from 0 to 2π, we get one basic cycle of the function.* Note that the basic sine cycle is 0 at its beginning, middle, and end points. The basic cosine function starts with $y = 1$, ends with $y = 1$, and $y = -1$ at the midpoint of the cycle.

Now consider what the graph of $y = \sin\left(x - \frac{\pi}{4}\right)$ should look like. We know that one basic cycle of the sine function is produced as the argument goes from 0 to 2π. In this case, the argument is the expression $x - \frac{\pi}{4}$. We

examine $x - \frac{\pi}{4}$ as it takes on all values from 0 to 2π to find out what values x takes on. We can do this algebraically, using

$$0 \leq x - \frac{\pi}{4} \leq 2\pi$$

This states that the argument runs (takes on values) from 0 to 2π. Now we can solve this statement for x by adding $\frac{\pi}{4}$ to each part.

$$\frac{\pi}{4} \leq x \leq 2\pi + \frac{\pi}{4}$$

$$\frac{\pi}{4} \leq x \leq \frac{9\pi}{4}$$

What we learn from this process is that for the expression $x - \frac{\pi}{4}$ to take on all values from 0 to 2π, x must take on all values from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$. Now we can reason as follows:

1. We know that one basic cycle of the sine function is produced as the argument, in this case $x - \frac{\pi}{4}$, varies from 0 to 2π.
2. The expression $x - \frac{\pi}{4}$ varies from 0 to 2π as x varies from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$.
3. Therefore, the expression $\sin\left(x - \frac{\pi}{4}\right)$ produces one basic cycle of the sine function as x varies from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$.

Thus, our basic cycle does not start at 0 and end at 2π but starts at $\frac{\pi}{4}$ and ends at $\frac{9\pi}{4}$. See figure 3–20.

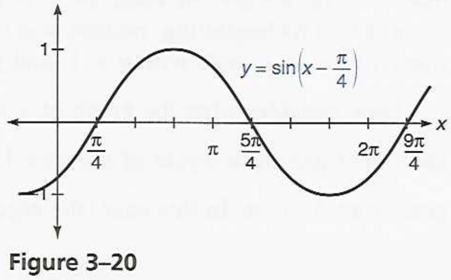
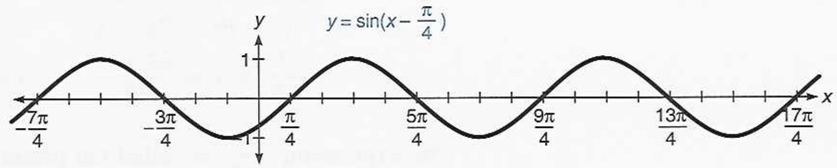


Figure 3–20

If we find the distance between $\frac{\pi}{4}$ and $\frac{9\pi}{4}$, it is $\frac{9\pi}{4} - \frac{\pi}{4} = \frac{8\pi}{4} = 2\pi$. This is the "length" of one basic cycle and is called the "period" of the function. We will define the period of the sine and cosine functions shortly. Right now, let us simply observe that a good rule for marking the x-axis is to divide it by using increments of one half of the period. To find this amount, divide the period by two (or multiply by one half). Thus, for convenience we marked the horizontal scale in increments of $\frac{2\pi}{2} = \pi$, starting at $\frac{\pi}{4}$. Note that the function crosses the x-axis halfway between the beginning and end of the cycle $\left(\text{at } \frac{5\pi}{4}\right)$ and has high and low points halfway between this point and the beginning and end points of the cycle.

Now, since we know that the sine function is periodic, and we have graphed one cycle, we repeat this cycle to obtain the complete graph. See figure 3–21.



$\boxed{\text{Y=}}$ $\boxed{\text{SIN}}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{-}$ $\boxed{\pi}$ $\boxed{\div}$ 4 $\boxed{)}$,

RANGE $-6,14,.785\left(\frac{\pi}{4}\right),-2,2,1$

Figure 3–21

Although we looked at the previous function in some detail, we can state the process we used in just a few steps. To understand these steps, remember the underlying idea we used:

> We know what the graphs of the sine and cosine functions look like as their arguments take on values from 0 to 2π. We therefore find out what values x has to take on for the argument to take on all values from 0 to 2π. As x takes on these values, we get one basic cycle of the sine or cosine function.

We now state the procedure to graph sine and cosine functions where the argument is of the form $Bx + C$, $B > 0$.

Graphing

$$y = A\sin(Bx + C) + D \text{ and}$$
$$y = A\cos(Bx + C) + D,\ B > 0$$

1. Solve $0 \le Bx + C \le 2\pi$ for x. This gives the left and right end points for one basic cycle.
2. Label the amplitude $|A|$. Use the left and right end points found in step 1, along with the amplitude, to draw one basic cycle. Reflect about the horizontal axis if $A < 0$.
3. Repeat this cycle to obtain as much of the graph as desired.
4. Apply a vertical shift D if necessary.

Note We will discuss the case where $B < 0$ shortly.

We need to define the term period, used above, and another term, phase shift, before proceeding with more examples. To do this we perform step 1 for the general case. Solving $0 \le Bx + C \le 2\pi$ for x:

$$0 \le Bx + C \le 2\pi$$
$$-C \le Bx \le 2\pi - C \qquad \text{Subtract } C \text{ from each expression}$$
$$-\frac{C}{B} \le x \le \frac{2\pi - C}{B} \qquad \text{Divide each expression by } B$$

The expression $-\dfrac{C}{B}$ is called the **phase shift** of the sine or cosine function being examined. The difference between the left and right end points of the basic cycle,

$$\begin{aligned}\frac{2\pi - C}{B} - \left(-\frac{C}{B}\right) &= \frac{2\pi - C}{B} + \frac{C}{B} \\ &= \frac{2\pi - C + C}{B} \\ &= \frac{2\pi}{B}\end{aligned}$$

is called the **period** of the sine or cosine function. B is also the number of complete cycles in 2π units.

It is not necessary to memorize these general expressions since the method we are using will produce these results anyway. With these terms, however, we can now state a precise *guideline for marking off the x-axis* when we graph. Mark the axis in increments of one half of the period, starting at the phase shift.

■ *Example 3–3 B*

1. Graph $y = 2 \cos 4x$. Show three cycles. State the amplitude, period, and phase shift.

Amplitude is 2, with no reflection about the x-axis.

Step 1: Solve $0 \le 4x \le 2\pi$ for x.

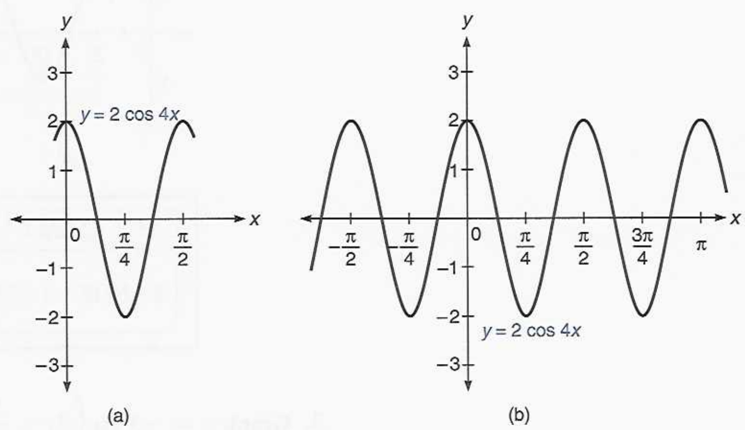
$$\frac{0}{4} \le \frac{4x}{4} \le \frac{2\pi}{4} \qquad \text{Divide by 4}$$

$$0 \le x \le \frac{\pi}{2}$$

We know that one basic cycle of the cosine function starts at 0 and ends at $\frac{\pi}{2}$. The period is $\frac{\pi}{2}$, and the phase shift is 0. The x-axis will be marked off in increments of one half of the period: $\frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$.

Step 2: Draw one basic cycle with amplitude 2. See part (a) of the figure.

Step 3: We get two more basic cycles by marking off one more period to the right of $\frac{\pi}{2}$, and one more to the left of 0. We then draw in the basic cycles. See part (b) of the figure.



$\boxed{\text{Y=}}$ 2 $\boxed{\text{COS}}$ 4 $\boxed{\text{X|T}}$, $\boxed{\text{RANGE } -2,4,.785,-3,3,1}$

2. Graph $y = \cos(3x - \pi)$. Show three cycles. State the amplitude, period, and phase shift.

Step 1: $0 \leq 3x - \pi \leq 2\pi$

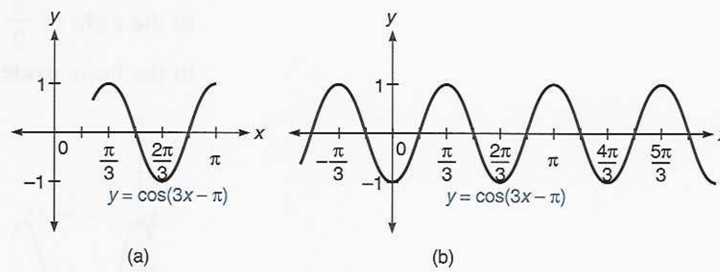
$\pi \leq 3x \leq 3\pi$ Add π to each expression

$\frac{\pi}{3} \leq x \leq \pi$ Divide each expression by 3

We know that we get one basic cycle of the cosine function as x varies from $\frac{\pi}{3}$ to π. The phase shift is $\frac{\pi}{3}$, and the period is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. We mark the x-axis in increments of $\frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$, starting at $\frac{\pi}{3}$, the phase shift, because this is where one basic cycle will start.

Step 2: We mark the amplitude, 1, and draw one basic cycle. See part (a) of the figure.

Step 3: In part (b) of the figure we show one more cycle on each ''side'' of the basic cycle.



Y=	COS	(3	X|T	−	π	),

RANGE $-1.25, 6, .524\left(\frac{\pi}{6}\right), -2, 2, 1$

3. Graph $y = -3\sin\left(3x - \frac{\pi}{4}\right)$. Show three cycles. State the amplitude, period, and phase shift.

The amplitude will be $|-3| = 3$ for this function. Because $-3 < 0$, there will be a reflection of the graph about the horizontal axis.

Step 1: $0 \leq 3x - \frac{\pi}{4} \leq 2\pi$

$0 \leq 12x - \pi \leq 8\pi$ Multiply each expression by 4

$\pi \leq 12x \leq 9\pi$

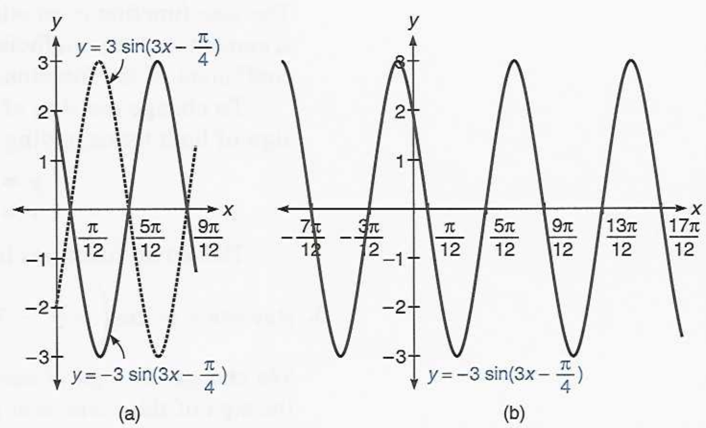
$\frac{\pi}{12} \leq x \leq \frac{9\pi}{12}$ or $\frac{\pi}{12} \leq x \leq \frac{3\pi}{4}$

The phase shift is $\frac{\pi}{12}$, and the period is $\frac{9\pi}{12} - \frac{\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$.

We mark the x-axis in increments of $\frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$, starting at $\frac{\pi}{12}$. Actually, we will use the value $\frac{4\pi}{12}$ instead of $\frac{\pi}{3}$ for convenience.

Step 2: We mark the x-axis in increments of $\frac{4\pi}{12}$ and draw a sine cycle between $\frac{\pi}{12}$ and $\frac{9\pi}{12}$ with amplitude 3. This is shown by the dashed lines in part (a) of the figure. Its reflection about the horizontal axis is shown in solid lines.

Step 3: We mark off more increments of $\frac{4\pi}{12}$ and draw two more cycles. See part (b) of the figure.



$\boxed{Y=}$ $\boxed{(-)}$ 3 $\boxed{SIN}$ $\boxed{(}$ 3 $\boxed{X|T}$ $\boxed{-}$ $\boxed{\pi}$ $\boxed{\div}$ 4 $\boxed{)}$,

$\boxed{\text{RANGE } -2,5,.262\left(\frac{\pi}{12}\right),-4,4,1}$

If B is negative in the argument $Bx + C$, we use the odd and even identities to get an equivalent expression with B positive. Recall these identities (section 3–1):

$$\sin(-x) = -(\sin x)$$
$$\cos(-x) = \cos x$$

Concept
We can change the sign of the argument of the sine function and still have an equivalent expression if we change the sign of the coefficient of the sine function itself. We can change the sign of the argument of the cosine function and still have an equivalent expression. We do *not* change the sign of the coefficient of the cosine function.

Example 3–3 C

1. Rewrite $y = 2\cos(-3x)$ so that the coefficient of x is positive.

   Changing the sign of the argument, $-3x$, we get $3x$. We do not change the sign of the coefficient of the cosine function since it is an even function. Thus,

   $$y = 2\cos(-3x) \text{ becomes}$$
   $$y = 2\cos 3x$$

   These are equivalent functions, so they have the same graph.

2. Rewrite $y = 3\sin(-2x + \pi)$ so that the coefficient of x is positive.

   The sine function is an odd function, so we change the sign of both the argument and the coefficient of the function. We change the sign of the coefficient of the function, 3, to -3.

   To change the sign of the argument, $-2x + \pi$, we must change the sign of both terms, giving $2x - \pi$. Thus,

   $$y = 3\sin(-2x + \pi) \text{ becomes}$$
   $$y = -3\sin(2x - \pi)$$

   These two functions have the same graph.

3. Rewrite $y = \cos\left(-\dfrac{x}{2} - 3\right)$ so that the coefficient of x is positive.

   We change the sign of each term of the argument, but we do not change the sign of the coefficient of the function itself. Thus,

   $$y = \cos\left(-\frac{x}{2} - 3\right) \text{ becomes}$$
   $$y = \cos\left(\frac{x}{2} + 3\right)$$

4. Rewrite $y = -\sin\left(-\dfrac{x}{3}\right)$ so that the coefficient of x is positive.

   Change the sign of $-\dfrac{x}{3}$ to $\dfrac{x}{3}$, and of the coefficient of the sine function, -1, to 1.

   $$y = -\sin\left(-\frac{x}{3}\right) \text{ becomes}$$
   $$y = \sin\frac{x}{3}$$

Example 3–3 D illustrates how to use the odd/even properties to help graph a function.

■ ***Example 3–3 D***

Graph $y = -2\sin\left(-\frac{2x}{3} + 1\right)$. Show three cycles, and state the amplitude, period, and phase shift.

Since the x term of the argument is negative, we use the odd property of the sine function to change both the sign of the argument and the sign of the coefficient of the function. This gives us the equivalent function

$$y = 2\sin\left(\frac{2x}{3} - 1\right)$$

Step 1: $0 \le \frac{2x}{3} - 1 \le 2\pi$

$$0 \le 2x - 3 \le 6\pi$$
$$3 \le 2x \le 6\pi + 3$$
$$\frac{3}{2} \le x \le \frac{6\pi + 3}{2}$$

Phase shift is $\frac{3}{2}$, and period is $\frac{6\pi + 3}{2} - \frac{3}{2} = \frac{6\pi}{2} = 3\pi$. We mark the x-axis by using $\frac{3}{2}$ as our starting point and using increments of $\frac{3\pi}{2}$, which is one half of the period. In this case, we also compute decimal approximations to the results to make it easier to plot our points. Several of the computations are

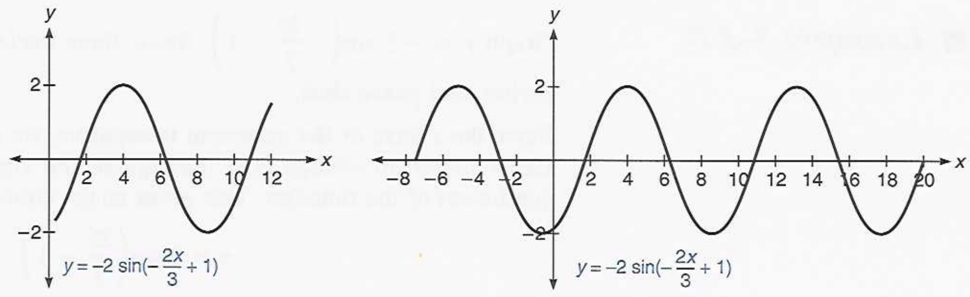
$$\frac{3}{2} + \frac{3\pi}{2} = \frac{3 + 3\pi}{2} \approx 6.2$$
$$\frac{3 + 3\pi}{2} + \frac{3\pi}{2} = \frac{3 + 6\pi}{2} \approx 10.9$$
$$\frac{3 + 6\pi}{2} + \frac{3\pi}{2} = \frac{3 + 9\pi}{2} \approx 15.6$$

and

$$\frac{3}{2} - \frac{3\pi}{2} = \frac{3 - 3\pi}{2} \approx -3.2$$
$$\frac{3 - 3\pi}{2} - \frac{3\pi}{2} = \frac{3 - 6\pi}{2} \approx -7.9$$

Step 2: One basic cycle is shown in part (a) of the figure.

Step 3: Two more cycles are shown in part (b) of the figure.



(a) (b)

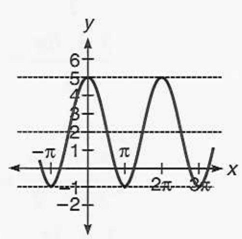
$\boxed{\text{Y=}}$ $\boxed{(-)}$ 2 $\boxed{\text{SIN}}$ $\boxed{(}$ $\boxed{(-)}$ 2 $\boxed{\text{X|T}}$ $\boxed{\div}$ 3 $\boxed{+}$ 1 $\boxed{)}$,
$\boxed{\text{RANGE } -8,20,2,-3,3,1}$

We can observe at this point that the method we have been using has given us both horizontal translations and scaling factors. Phase shift is a horizontal translation, and if we divide the period, $\frac{2\pi}{B}$, by 2π (the period of the basic sine or cosine function), we get $\frac{1}{B}$, a horizontal scale factor. Normally we do not actually compute the horizontal scale factor.

There are times when we will want to find the equation of a function, given some of its properties.

Example 3–3 E

1. Find the equation of the cosine function in the figure.

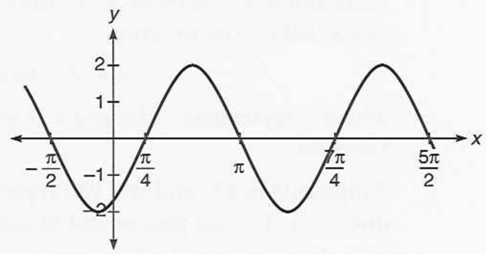


We know the equation is of the form

$$y = A\cos(Bx + C) + D$$

It is shifted up 2 units, so D is $+2$. The distance between the high and low points is 6. The amplitude is one half this value, or 3, so $|A| = 3$. Since a basic cycle starts at 0 and ends at 2π, we know the argument is simply x. Thus, the equation is $y = 3\cos x + 2$.

2. Find an equation of the sine function in the figure.



We know that the equation is of the form

$$y = A\sin(Bx + C) + D$$

Since the distance between the high and low points of the graph is 4, we know that A is 2. Also, there is no vertical shift, so D is 0.

We need to find the argument of the function. Note that a cycle of this function starts at $\frac{\pi}{4}$ and ends at $\frac{7\pi}{4}$. We can work backward from this information.

We know that we get one basic cycle as x takes on values between these two points; that is, $\frac{\pi}{4} \le x \le \frac{7\pi}{4}$.

Our objective is to arrange the values so that the left value is 0 and the right value is 2π. (Remember, we are working back to the argument of the function.)

First, we want the left value to be 0.

$$\pi \le 4x \le 7\pi \qquad \text{Multiply by 4}$$
$$0 \le 4x - \pi \le 6\pi \qquad \text{Subtract } \pi$$

Now we want the right point to be 2π. Dividing by 3 will do this.

$$\frac{0}{3} \le \frac{4x - \pi}{3} \le \frac{6\pi}{3}$$

$$0 \le \frac{4x}{3} - \frac{\pi}{3} \le 2\pi$$

We thus find the argument is $\frac{4x}{3} - \frac{\pi}{3}$. Thus, our final answer is $y = 2\sin\left(\frac{4x}{3} - \frac{\pi}{3}\right)$.

In some applications we want to express values of x in degrees as opposed to radians. Our procedures are the same, except that our limits for the basic cycles are 0° and 360° instead of 0 and 2π.

■ ***Example 3–3 F***

1. An AC (alternating current) signal with peak-to-peak voltage of 170 volts and phase shift of 120°, riding on a DC level of 100 volts, could be described by the function

$$y = 85 \sin(x + 120°) + 100$$

where y represents volts and x is in degrees. Graph one cycle of this function.

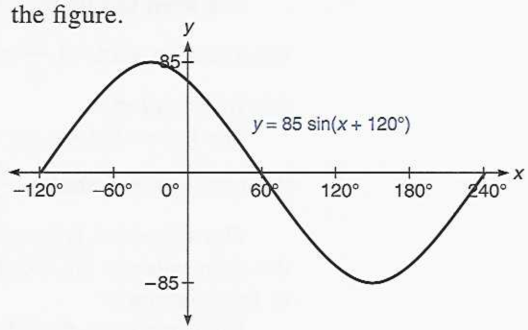
Amplitude is 85, and the 100 represents a vertical shift in the positive direction. To find period and phase shift we proceed as follows:

Step 1: $0° \leq x + 120° \leq 360°$

$-120° \leq x \leq 240°$

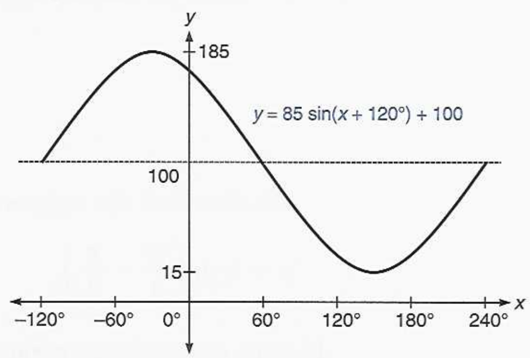
Phase shift is $-120°$ and period is $240° - (-120°) = 360°$. We mark the x-axis in increments of half the period, 180°, starting at $-120°$.

Step 2: We see that a basic cycle begins at $-120°$ and ends at 240°. See the figure.



(a)

Step 3: We also label the amplitude, 85. We then shift the graph vertically by 100 units. See the figure.



(b)

Put the calculator in DEGREE mode (use the $\boxed{\text{MODE}}$ key).
$\boxed{\text{Y=}}$ 85 $\boxed{\text{SIN}}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 120 $\boxed{)}$ $\boxed{+}$ 100 $\boxed{\text{ENTER}}$ 100,
$\boxed{\text{RANGE } -130{,}250{,}60{,}-10{,}190{,}20}$

2. An electronic signal is to be modeled with the sine function. The peak-to-peak voltage is 340 volts (amplitude is 170 volts). There is a phase shift of 30°, and the period is 120°. The signal is at 200 volts above ground potential. (There is a vertical shift of 200.) Find the sine function that will model this signal.

We know that the function is of the form

$$y = A\sin(Bx + C) + D$$

and that $A = 170$ and $D = 200$. To find B and C we can proceed "backward." We know that we get one basic cycle as x varies between 30°, the phase shift, and $30° + 120°$, or phase shift + period.

$$30° \le x \le 30° + 120°$$

Now adjust this so that phase shift is 0° and period is 360°.

$$30° \le x \le 150°$$

Subtract 30° from each expression to get 0° phase shift.

$$0° \le x - 30° \le 120°$$

Multiply each term by 3, since this will make the end point 360°.

$$0° \le 3x - 90° \le 360°$$

Thus, the argument of the function is $3x - 90°$, and the function we want is

$$y = 170\sin(3x - 90°) + 200$$ ■

Mastery points

Can you

- Graph an equation of the form

$$y = A\sin(Bx + C) + D \text{ or}$$
$$y = A\cos(Bx + C) + D?$$

- Find a sine or cosine equation that is appropriate, given values of A and D and the initial and terminal points of a basic cycle?

Exercise 3–3

Graph three cycles of the following functions.

1. $y = 5\sin x$
2. $y = 5\cos x$
3. $y = \frac{2}{3}\cos x$
4. $y = \frac{1}{5}\sin x$
5. $y = -4\cos x$
6. $y = -2\sin x$
7. $y = -\frac{1}{3}\sin x$
8. $y = -\frac{5}{2}\sin x$
9. $y = 2\sin x + 1$
10. $y = 3\cos x - 2$
11. $y = -\frac{3}{4}\cos x - 2$
12. $y = -\frac{1}{2}\sin x + 3$

Graph three cycles of the following functions. State the amplitude, period, and phase shift of each.

13. $y = 2\sin 4x$ **14.** $y = 3\cos\dfrac{x}{2}$ **15.** $y = \cos\left(x - \dfrac{\pi}{2}\right)$ **16.** $y = 3\sin(2x + \pi)$

17. $y = \frac{2}{3}\sin(3x + \pi)$ **18.** $y = \frac{5}{8}\cos 5x$ **19.** $y = -\cos 3x$ **20.** $y = -\sin x$

21. $y = -\cos\left(2x + \dfrac{\pi}{2}\right)$ **22.** $y = -\sin\left(3x - \dfrac{\pi}{3}\right)$ **23.** $y = \sin(3x + 2\pi)$ **24.** $y = \cos(2x - 3\pi)$

25. $y = \cos 2\pi x$ **26.** $y = \sin \pi x$ **27.** $y = 2\sin 3x + 2$ **28.** $y = 3\cos 2x - 3$

29. $y = -3\cos x + 1$ **30.** $y = -\sin 4x + 1$ **31.** $y = 2\sin(2x - \pi) + 1$ **32.** $y = 3\sin(3x + \pi) - 3$

33. $y = \sin \pi x + 1$ **34.** $y = 2\cos\dfrac{\pi x}{2} - 2$

Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive.

35. $y = \sin(-2x)$ **36.** $y = \cos(-x)$ **37.** $y = -\cos(-3x)$ **38.** $y = -\sin(-5x)$

39. $y = \sin(-x - 3)$ **40.** $y = \cos(-2x + 4)$ **41.** $y = \sin(-x) - 3$ **42.** $y = \cos(-2x) + 4$

43. $y = -3\cos\left(-2x + \dfrac{\pi}{2}\right)$ **44.** $y = 2\sin\left(-\dfrac{x}{3} - \pi\right)$

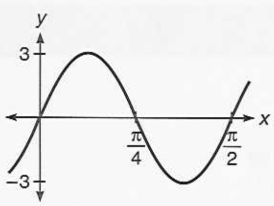
Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph three cycles of the function.

45. $y = \sin(-x)$ **46.** $y = \cos(-2x)$ **47.** $y = \cos\left(-x - \dfrac{\pi}{3}\right)$ **48.** $y = 2\sin(-2x + \pi)$

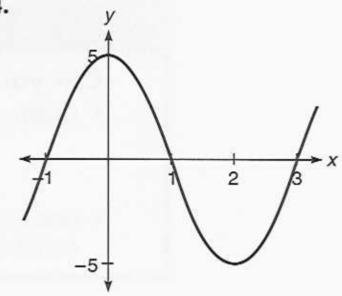
49. $y = -\sin(-2\pi x + \pi)$ **50.** $y = -\cos(-\pi x)$ **51.** $y = \sin(-\pi x + 1)$ **52.** $y = 2\cos(-3\pi x - 2)$

Assume that each of the following graphs is the graph of a sine function of the form $y = A\sin(Bx + C) + D$. Find values of A, B, C, and D that would produce each graph.

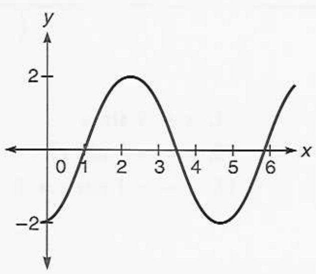
53.



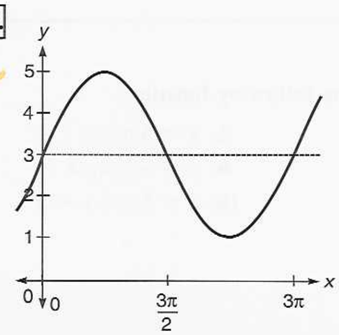
54.



55.



56.



57. Do problem 53, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

58. Do problem 54, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

59. Do problem 55, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

60. Do problem 56, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

Graph one cycle of each of the following functions. Mark the horizontal axis in degrees.

61. $y = 3\sin(x + 60°)$ **62.** $y = -50\cos(x - 120°)$ **63.** $y = 25\cos 3x$ **64.** $y = 10\sin(2x - 180°)$

65. An electronic signal modeled with the sine function has a peak-to-peak voltage of 120 volts (amplitude is 60 volts), phase shift of 90°, and period of 54°. Find an equation of the sine function that will model this signal.

66. An ocean wave is being modeled with the sine function. Its amplitude is 6 feet and its phase shift (with respect to another wave) is −180°. If the period is 720°, find an equation of the sine function that will model this wave.

67. One of the components of a function that could describe the earth's ice ages for the last 500,000 years is described by a sine function with amplitude 0.5, period $\frac{360°}{43}$, 0° phase shift, and vertical translation 23.5. Find an equation for this component.

68. The activity of sunspots seems to follow an 11-year cycle. Assuming that this activity can be roughly modeled with a sine wave, construct a sine function with period $\frac{360°}{11}$, amplitude 1, phase shift 90°, and vertical translation 2.

69. Graph the following functions on the same set of axes: $y = \sin x$; $y = \sin 3x$; and $y = \sin\frac{x}{3}$.

70. Graph the following functions on the same set of axes: $y = \sin x$; $y = \sin\left(x + \frac{\pi}{2}\right)$; and $y = \sin x + \frac{\pi}{2}$.

71. Graph the following functions on the same set of axes: $y = \cos\left(\frac{\pi}{2} - x\right)$ and $y = \sin x$. Draw a conclusion from the graph. $\left[\textit{Hint:} \text{ Rewrite as } y = \cos\left(-x + \frac{\pi}{2}\right).\right]$

3–4 Linear transformations of the tangent, cotangent, secant, and cosecant functions (optional)

The tangent and cotangent functions

The tangent and cotangent functions are π-periodic, so the basic cycle for each is π units long instead of the 2π units for the sine and cosine functions. Figure 3–22 shows a basic cycle for the tangent and cotangent functions.

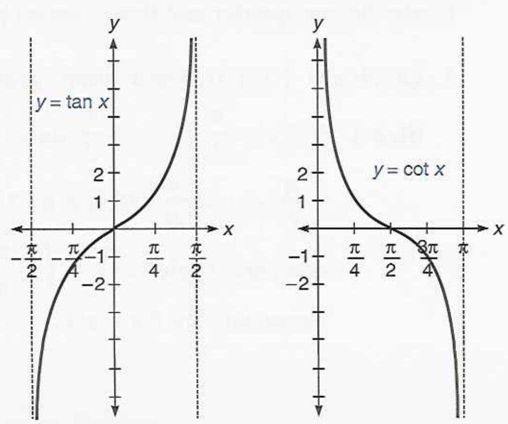


Figure 3–22

Note that the basic tangent cycle starts at $-\frac{\pi}{2}$ and ends at $\frac{\pi}{2}$. The basic cotangent cycle starts at 0 and ends at π. Also note that the functions are $+1$ or -1 at the points that are $\frac{1}{4}$ and $\frac{3}{4}$ of the distance between the cycle end points. We will call these the *one-quarter* and *three-quarter* points.

Graphing functions of the form $y = A\tan(Bx + C)$ and $y = A\cot(Bx + C)$ is done in a manner very similar to that for the sine and cosine functions. Although the concept of amplitude does not make sense for these functions, the vertical scaling factor A does affect the graph. In fact, these functions take on values of $\pm A$ at the one-quarter and three-quarter points (unless there is a vertical shift) instead of ± 1.

To graph functions of the form

$$y = A\tan(Bx + C) \text{ and}$$
$$y = A\cot(Bx + C)$$

1. For the tangent function, solve

$$-\frac{\pi}{2} < Bx + C < \frac{\pi}{2}$$

for x.
For the cotangent function, solve

$$0 < Bx + C < \pi$$

for x.
2. Step 1 gives the left and right end points for one basic cycle. Draw this cycle. Label the "one-quarter" and "three-quarter" points with $y = A$ and $y = -A$, as appropriate. If $A < 0$, this cycle is reflected about the horizontal axis.
3. Repeat this cycle to obtain as much of the graph as desired.

We will not concern ourselves with defining the period and phase shift for the tangent and cotangent functions. A *guideline* for marking off the x-axis is to use increments of one fourth of the length of one basic cycle to locate the one-quarter and three-quarter points.

Example 3–4 A

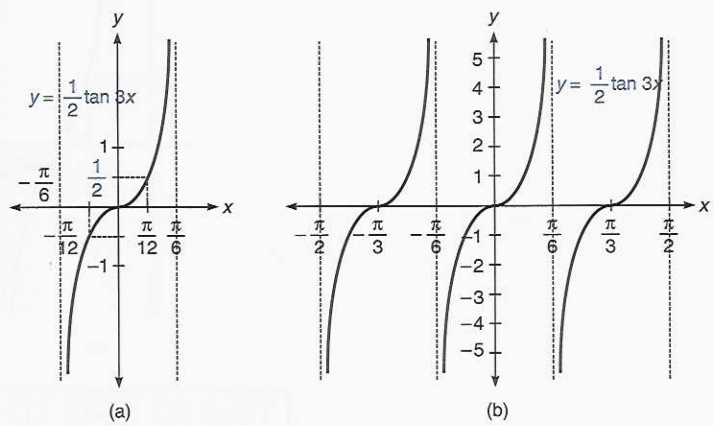
1. Graph $y = \frac{1}{2}\tan 3x$. Show three cycles.

Step 1: Solve $-\frac{\pi}{2} < 3x < \frac{\pi}{2}$ for x.

$-\frac{\pi}{6} < x < \frac{\pi}{6}$. Divide by 3 (or multiply by $\frac{1}{3}$). The length of one basic cycle is $\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$. We use $\frac{1}{4} \cdot \frac{\pi}{3} = \frac{\pi}{12}$ increments on the x-axis.

Step 2: We now know that we get one basic tangent cycle starting at $-\dfrac{\pi}{6}$ and ending at $\dfrac{\pi}{6}$. We draw vertical asymptotes at these points and sketch one cycle of the tangent function. With the vertical scaling factor of $\frac{1}{2}$ we label the one-quarter and three-quarter points as shown in part (a) of the figure.

Step 3: Two more cycles are shown in part (b) of the figure.



$\boxed{\text{Y=}}$ .5 $\boxed{\text{TAN}}$ 3 $\boxed{\text{X}\mid\text{T}}$,
RANGE $-2,2,.524\left(\dfrac{\pi}{6}\right),-4,4,1$

2. Graph $y = \cot\left(2x - \dfrac{\pi}{3}\right)$. Show three cycles.

Step 1: We put the argument between 0 and π and solve for x:

$$0 < 2x - \frac{\pi}{3} < \pi$$

$$0 < 6x - \pi < 3\pi \qquad \text{Multiply by 3}$$

$$\pi < 6x < 4\pi \qquad \text{Add } \pi$$

$$\frac{\pi}{6} < x < \frac{4\pi}{6} \qquad \text{Divide by 6}$$

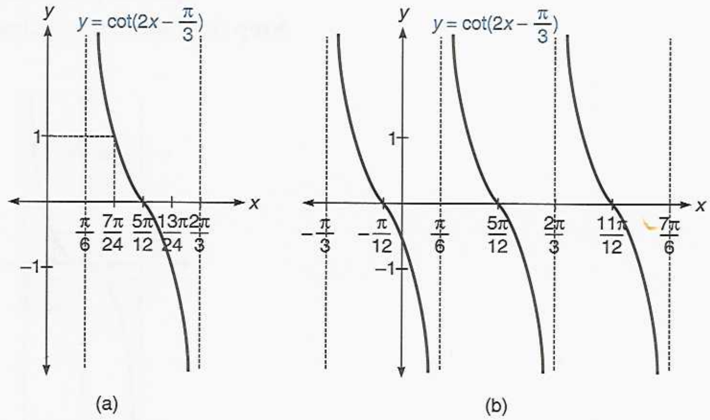
or

$$\frac{\pi}{6} < x < \frac{2\pi}{3}$$

A basic cycle is $\dfrac{2\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{2}$ units long. We mark the x-axis by adding or subtracting increments of $\dfrac{\pi}{8}$ units from $\dfrac{\pi}{6}$.

Step 2: We have one basic cycle between $\frac{\pi}{6}$ and $\frac{2\pi}{3}$; the basic graph is shown in part (a) of the figure.

Step 3: The finished graph, including three cycles, is shown in part (b) of the figure.



$\boxed{\text{Y=}}$ $\boxed{(}$ $\boxed{\text{TAN}}$ $\boxed{(}$ 2 $\boxed{\text{X|T}}$ $\boxed{-}$ $\boxed{\pi}$ $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{)}$ $\boxed{x^{-1}}$,
$\boxed{\text{RANGE } -1.5,4,0.524,-3,3,1}$

If the coefficient of x is negative, we use the fact that the tangent and cotangent functions are odd to rewrite the function with this coefficient positive.

■ ***Example 3–4 B***

Graph $y = -\cot(-\pi x)$. Show three cycles.

Since the coefficient of x, $-\pi$, is negative, we rewrite this function as

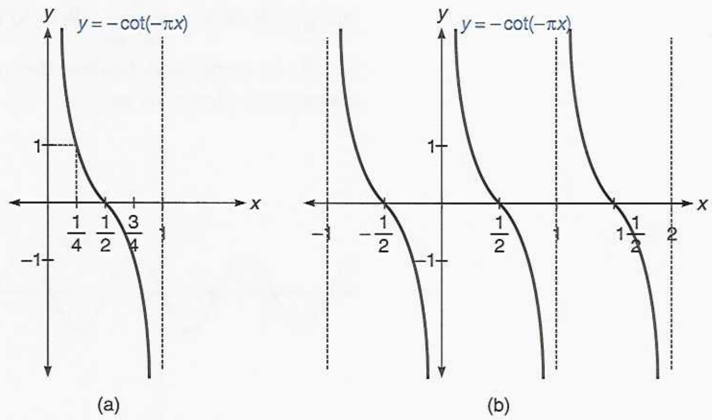
$$y = \cot \pi x.$$

Step 1: $0 < \pi x < \pi$
$0 < x < 1$ Divide by π

A basic cycle is 1 unit long, and we use $\frac{1}{4}$ for an increment on the x-axis.

Step 2: One basic cycle is shown in part (a) of the figure.

Step 3: Two more cycles are shown in part (b) of the figure.



| Y= | (−) | (| TAN | (−) | π | X\|T |) | x^{-1} |, RANGE −1.2,2.2,.5,−3,3,1

The secant and cosecant functions

To graph variations of the secant and cosecant functions, we use the fact that they are reciprocals of the cosine and sine functions, respectively. Figure 3–11 shows the graph of the cosecant function and figure 3–12 shows the graph of the secant function. Observe that the ranges are $|y| \geq 1$, and that they have vertical asymptotes where their reciprocal function, cosine or sine, is zero.

Consider the graph of a function of the form

$$y = A \csc(Bx + C)$$

We know it is a modification of the graph shown in figure 3–11. Since it is equivalent to the graph of

$$y = A\left(\frac{1}{\sin(Bx + C)}\right)$$

we can construct the graph of $y = \sin(Bx + C)$ and, graphically, form the reciprocal to get the graph we want.

For example, consider the graph of $y = 3 \csc 2x$. This will have the same graph as $y = 3\left(\frac{1}{\sin 2x}\right)$. Thus, we can first graph $y = \sin 2x$ and graphically form the reciprocal, as we did in section 3–3.

The graph of $y = \sin 2x$ is shown in figure 3–23. In figure 3–24 we show the graph of $y = \dfrac{1}{\sin 2x}$. We do this by drawing vertical asymptotes wherever $\sin 2x$ is zero, and letting the reciprocal graph go to infinity as $\sin 2x$ gets closer and closer to zero.

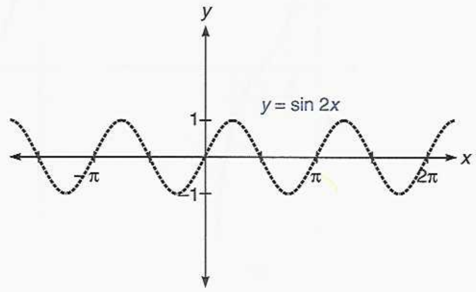


Figure 3–23

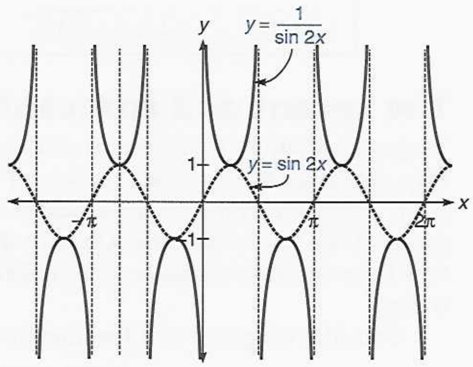
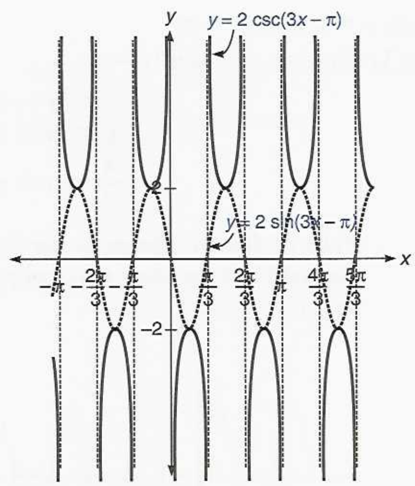


Figure 3–24

In figure 3–25 we show the graph of $y = 3\left(\dfrac{1}{\sin 2x}\right)$. Each point is three times higher or lower than each corresponding point on the graph in figure 3–25. This is also the graph we wanted originally.



$\boxed{\text{Y=}}$ 3 $\boxed{(}$ $\boxed{\text{SIN}}$ 2 $\boxed{\text{X|T}}$ $\boxed{)}$ $\boxed{x^{-1}}$,

$\boxed{\text{RANGE } -4,7,1.57,-5,5,1}$

Figure 3–25

Observe that we could have originally graphed $y = 3 \sin 2x$ and used this as our guide, since in the graph of $y = 3 \sin 2x$, $|y| \leq 3$, while in the graph of $y = 3 \csc 2x$, $|y| \geq 3$.

Based on this example, and what we have done in the preceding sections of this chapter, we can state the following.

Procedure for graphing functions of the form

$$y = A \csc(Bx + C) \text{ and}$$
$$y = A \sec(Bx + C)$$

1. Graph $y = A \sin(Bx + C)$ or $y = A \cos(Bx + C)$, whichever is the appropriate reciprocal function.
2. Graphically form the reciprocal by drawing vertical asymptotes wherever the graph in step 1 is 0, and draw the graph getting larger and larger in absolute value wherever the reciprocal function approaches 0.

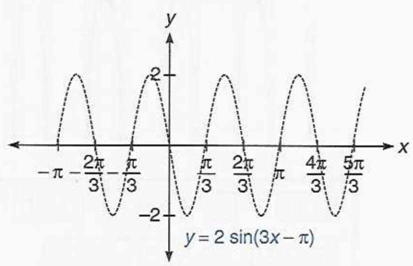
■ ***Example 3–4 C***

Graph $y = 2\csc(3x - \pi)$.

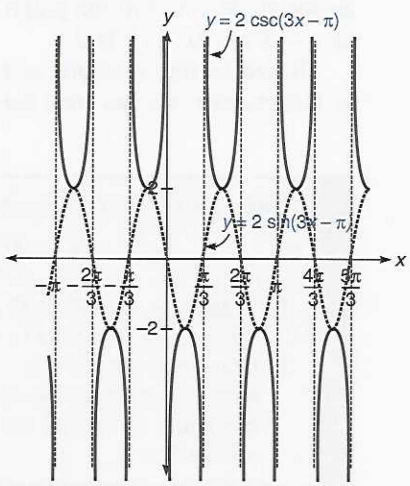
Step 1: Graph $y = 2\sin(3x - \pi)$.

$$0 \le 3x - \pi \le 2\pi$$
$$\pi \le 3x \le 3\pi$$
$$\frac{\pi}{3} \le x \le \pi$$

Four cycles are shown in the first figure. We draw this graph using dashed lines, because it is not part of the graph we have been asked to show.



Step 2: We draw vertical asymptotes wherever the graph of $y = 2\sin(3x - \pi)$ is 0; that is, where it crosses the x-axis. We then form the reciprocal function, as shown in the next figure.



Y=	2	(	SIN	(	3	X|T	−	π	)	)	x^{-1},

RANGE $-3.2, 6, 1.05\left(\frac{\pi}{3}\right), -4, 4, 1$

If the coefficient of x is negative, we use the odd/even properties of the sine and cosine functions as appropriate.

■ ***Example 3–4 D*** Graph $y = 2\csc\left(-\frac{2\pi x}{3}\right)$.

Step 1: Graph $y = 2\sin\left(-\frac{2\pi x}{3}\right)$. Since the argument is negative, we use the odd property of the sine function to rewrite this as $y = -2\sin\frac{2\pi x}{3}$.

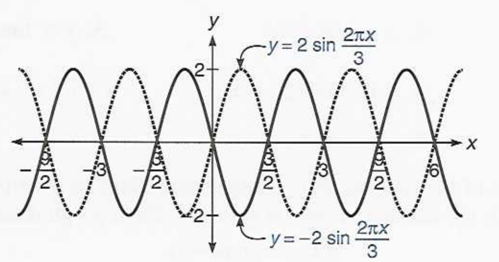
$$0 \le \frac{2\pi x}{3} \le 2\pi$$

$$0 \le 2\pi x \le 6\pi \quad \text{Multiply by 3}$$

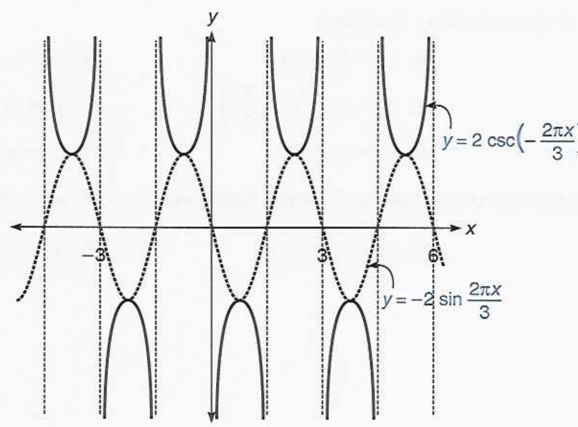
$$0 \le x \le 3 \quad \text{Divide by } 2\pi$$

The graph of $y = -2\sin\frac{2\pi x}{3}$ is shown in the figure, part (a).

Step 2: The reciprocal function is shown in part (b) of the figure.



(a)



(b)

| Y= | 2 | (| SIN | (| (−) | 2 | π | X\|T | ÷ | 3 |) |) | x^{-1} |,
RANGE −4.5,6.5,1.5,−4,4,1 |

■

Mastery points

Can you

- Graph functions of the form

$$y = A\tan(Bx + C), \text{ and}$$
$$y = A\cot(Bx + C)?$$

- Graph functions of the form

$$y = A\sec(Bx + C) \text{ and}$$
$$y = A\csc(Bx + C)?$$

Exercise 3–4

Graph three cycles of the following functions.

1. $y = 5\tan x$ **2.** $y = -4\cot x$ **3.** $y = \tan 4x$ **4.** $y = \cot\frac{x}{2}$

5. $y = \cot\left(x - \frac{\pi}{2}\right)$ **6.** $y = 3\tan(2x + \pi)$ $\boxed{7.}$ $y = -\cot\left(2x + \frac{\pi}{2}\right)$ **8.** $y = -\tan\left(3x - \frac{\pi}{3}\right)$

9. $y = \cot 2\pi x$ **10.** $y = \tan \pi x$

Use the odd/even properties of the tangent and cotangent functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph three cycles of the function.

11. $y = \tan(-2x)$ **12.** $y = \cot(-x)$ **13.** $y = -\cot(-\pi x)$

14. $y = -\tan(-2\pi x)$ $\boxed{15.}$ $y = \tan(-x - \pi)$ **16.** $y = \cot(-2x + 4\pi)$

Graph three cycles of the following functions.

17. $y = \frac{2}{3}\csc x$ **18.** $y = \frac{1}{5}\sec x$ **19.** $y = -4\csc x$ **20.** $y = 2\sec 4x$

21. $y = 3\csc\frac{x}{2}$ **22.** $y = \csc\left(x - \frac{\pi}{2}\right)$ 23. $y = 3\sec(2x + \pi)$ $\boxed{24.}$ $y = \frac{2}{3}\sec(3x + \pi)$

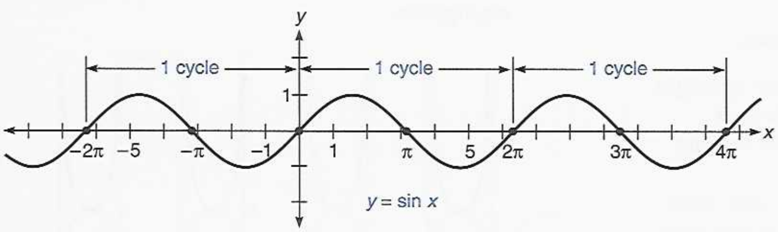
25. $y = \csc(2x - 3\pi)$ **26.** $y = \csc 2\pi x$ **27.** $y = \sec \pi x$

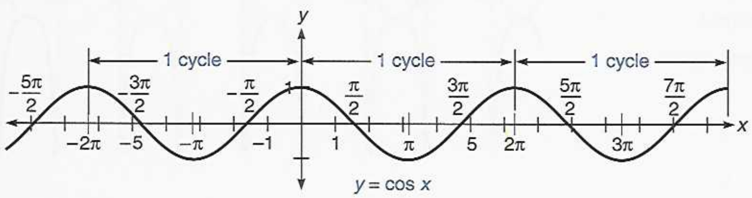
Use the odd/even properties of the sine and cosine functions to graph each of the following functions.

$\boxed{28.}$ $y = \sec(-2x)$ **29.** $y = \csc(-x)$ **30.** $y = 3\csc\left(-2x + \frac{\pi}{2}\right)$ 31. $y = 2\sec\left(-\frac{x}{3} - \pi\right)$

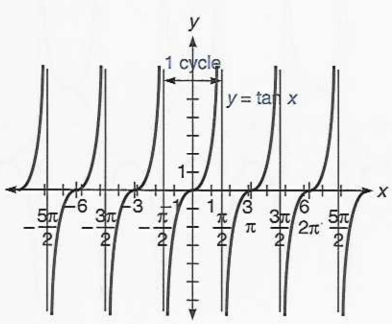
Chapter 3 summary

- **Basic graphs**
- Graph of the sine and cosine functions.





- Graph of the tangent function.

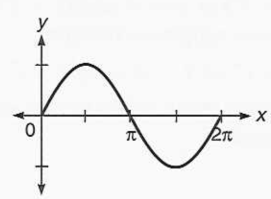


- To graph sine and cosine functions of the form

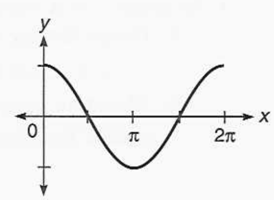
  $y = A\sin(Bx + C) + D$ and
  $y = A\cos(Bx + C) + D$, where $B > 0$

1. Solve $0 \leq Bx + C \leq 2\pi$ so x is the middle member.
   - This gives the left and right end points for one basic cycle.
   - The left end point is the phase shift.
   - The difference between the end points is the period.
2. The amplitude is $|A|$.
   - This is the height of the basic graph above and below the x-axis.
   - The graph is shifted about the horizontal axis if $A < 0$.

   Draw one basic cycle with the information from steps 1 and 2.
3. Repeat the cycle obtained from steps 1 to 3 to obtain more of the graph.
4. Shift the graph vertically D units.



Basic sine cycle



Basic cosine cycle

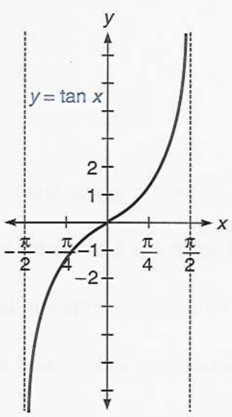
- If the coefficient of x, B, is negative in the argument $Bx + C$ we first use the odd and even properties to get an equivalent expression with B positive.

- To graph functions of the form

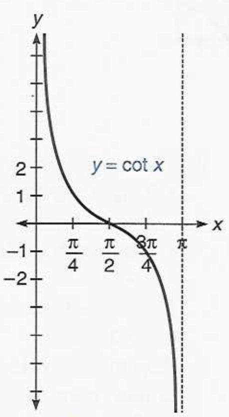
$$y = A\tan(Bx + C) \text{ and}$$
$$y = A\cot(Bx + C),\ B > 0$$

1. For the tangent function, solve $-\frac{\pi}{2} < Bx + C < \frac{\pi}{2}$ for x; for the cotangent function solve $0 < Bx + C < \pi$ for x. This gives the left and right end points for one basic cycle. The difference between the end points is the period. The left end point is the phase shift.
2. Use the values from step 1 to draw one basic cycle. Label the one-quarter and three-quarter points with $y = A$ and $y = -A$ as appropriate. Repeat this cycle to obtain as much of the graph as desired.

Basic tangent cycle



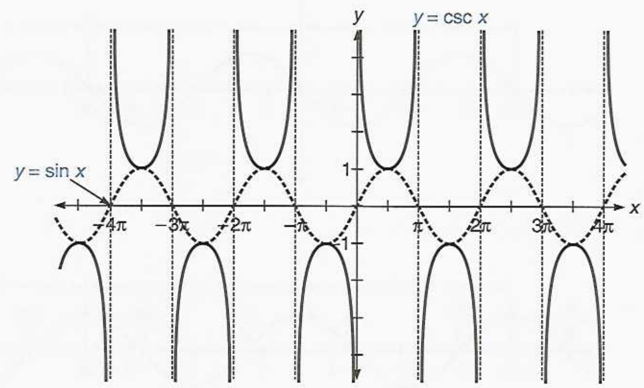
Basic cotangent cycle

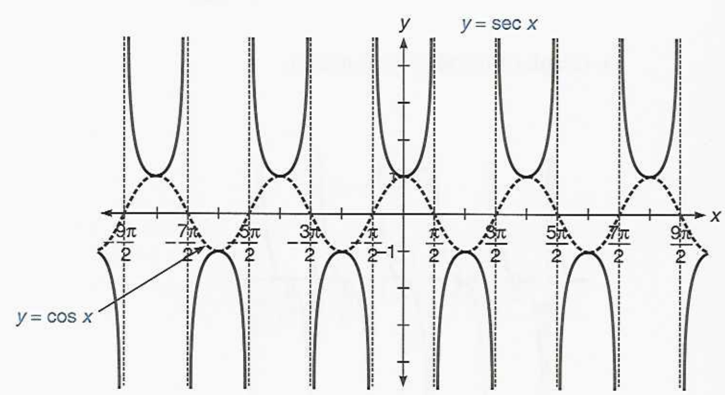


- To graph $y = A\csc(Bx + C)$ or $y = A\sec(Bx + C)$:
1. Graph the appropriate reciprocal function,

$$y = A\sin(Bx + C) \text{ or } y = A\cos(Bx + C)$$

2. Sketch in vertical asymptotes wherever the sine or cosine function is zero.
3. Create the cosecant or secant graph by starting at the highest and lowest points of the sine or cosine graph and sketching values that increase in absolute value from that point as x approaches the vertical asymptotes. Note that these functions are not defined at the asymptotes.





- Summary of the properties of the sine, cosine, and tangent functions (k an integer).

Function	Domain	Range	Period
$y = \sin x$	R	$-1 \le y \le 1$	2π
$y = \cos x$	R	$-1 \le y \le 1$	2π
$y = \tan x$	$x \ne \frac{\pi}{2} + k\pi$	R	π

$\sin(-x) = -\sin x$ (odd)
$\cos(-x) = \cos x$ (even)
$\tan(-x) = -\tan x$ (odd)

- Summary of the properties of the cosecant, secant, and cotangent functions (k an integer).

Function	Domain	Range	Period
$y = \csc x$	$x \ne k\pi$	$\lvert y\rvert \ge 1$	2π
$y = \sec x$	$x \ne \frac{\pi}{2} + k\pi$	$\lvert y\rvert \ge 1$	2π
$y = \cot x$	$x \ne k\pi$	R	π

$\csc(-x) = -\csc x$ (odd)
$\sec(-x) = \sec x$ (even)
$\cot(-x) = -\cot x$ (odd)

Chapter 3 review

[3–1]

1. Sketch the graph of the sine function; state the domain, range, and period of the sine function.

2. Using the graph of $y = \cos x$ as a guide, describe all values of x for which $\cos x$ is 1.

Use the appropriate property, even or odd, to calculate the exact function value.

3. $\cos\left(-\frac{\pi}{6}\right)$ **4.** $\tan\left(-\frac{4\pi}{3}\right)$

5. $\sin\left(-\frac{5\pi}{6}\right)$ **6.** $\sec\left(-\frac{\pi}{4}\right)$

Test the function for the even/odd property.

7. $f(x) = \frac{x^2 - 1}{x}$ **8.** $f(x) = x \sin x$

9. $f(x) = \tan x \cdot \cos x$

[3–2]

10. Sketch the graph of the cosecant function.

11. State the domain and range of the cotangent function.

12. Show that $f(x) = \frac{x}{\sec x}$ is an odd function.

13. Show that the function $f(x) = \sec x \cdot \sin^2 x + x^4$ is an even function.

[3–3] Graph three cycles of the following functions. State the amplitude, period, and phase shift of each.

14. $y = 2 \sin x$ **15.** $y = -\frac{2}{3} \cos x$

16. $y = 3 \sin x - 2$ **17.** $y = 2 \sin 3x$

18. $y = \cos\left(x + \frac{\pi}{3}\right)$ **19.** $y = 2 \sin\left(\frac{x}{2} + \frac{\pi}{3}\right)$

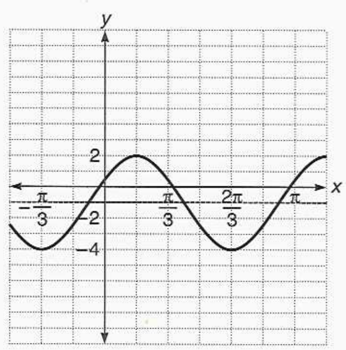
20. $y = \cos 3x\pi$ **21.** $y = 3 \cos 2x - 3$

Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph three cycles of the function.

22. $y = \cos\left(-2x + \frac{\pi}{2}\right)$ **23.** $y = 3 \sin(-x + \pi)$

24. Graph three cycles of the function $y = 2 \sin(3x + 60°)$. Mark the horizontal axis in degrees.

25. Assume that the following graph is of a cosine function of the form $y = A \cos(Bx + C) + D$. Find the values of A, B, C, and D and rewrite the function using these values.



26. Most people have heard about the theory of biorhythms. This theory maintains that at birth three cycles are started—physical, emotional, and intellectual. The physical cycle has a period of 23 (days). Assuming an amplitude of 1, a phase shift of -10 (days), and no vertical translation, create an equation that describes the physical cycle in terms of the sine function.

[3–4] Graph three cycles of the following functions.

27. $y = \tan 4x$

28. $y = \tan(3x + \pi)$

29. $y = 2\cot\left(x - \dfrac{\pi}{4}\right)$

30. $y = -2\sec 3x$

31. $y = \csc \dfrac{x}{2}$

32. $y = \sec(2x - \pi)$

33. $y = \csc 3\pi x$

Chapter 3 test

1. Using the graph of $y = \sin x$ as a guide, describe all values of x for which $\sin x$ is -1.

2. Use the appropriate property, even or odd, to calculate the exact function value of $\tan\left(-\dfrac{5\pi}{3}\right)$.

3. Test the function $f(x) = x + \sin x$ for the even/odd property.

Graph three cycles of the following functions. State the amplitude, period, and phase shift of each.

4. $y = 2\sin x + 2$

5. $y = 3\cos 2x$

6. $y = 3\sin\left(\dfrac{x}{3} + \dfrac{\pi}{2}\right)$

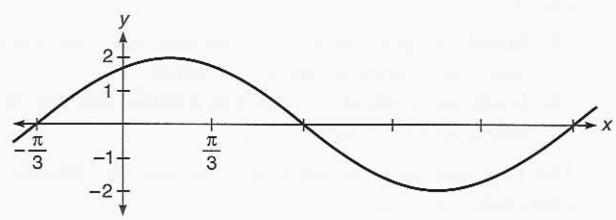
Graph three cycles of the following functions.

7. $y = 3\tan \pi x$

8. $y = \sec(3x - \pi)$

9. $y = -\csc 4\pi x$

10. Assume that the graph is of the form $y = A\sin(Bx + C) + D$. Find values of A, B, C, and D that would produce the graph and write the corresponding equation.



11. Sketch the graph of the secant function.

12. Show that the function $f(x) = \sec x \cdot \sin x + x^3$ is an odd function.

13. In the theory of biorhythms, the emotional cycle has a period of 28 (days). Assuming an amplitude of 1, a phase shift of 5 (days), and no vertical translation, create an equation that describes the emotional cycle in terms of the sine function.

14. An electronic signal is to be modeled with the sine function. Amplitude is 25 volts, phase shift is $-20°$, period is 150°, and there is a vertical shift of 10 volts. Find a sine function that will model this signal.

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